6 Supplemental calculations (not for publication)

This supplement provides details of some of the calculations used in the proof of Theorem 1. Equation and page numbering continues in sequence with that of the main text.

The structure of the $(K+L)\times(K+L)$ $\nabla_{\mathbf{r}_0}\mathbf{B}(\mathbf{r}_0)=\nabla_{\mathbf{r}_0}\mathbf{B}(\mathbf{r}_0;\mathbf{p},\mathbf{q},\theta)$ matrix is

$$
\begin{aligned}
\nabla_{\mathbf{r}_0}\mathbf{B}(\mathbf{r}_0) &= \begin{pmatrix}
\nabla_{r_{10}}B_{10} & \cdots & \nabla_{r_{K0}}B_{10} & \nabla_{r_{01}}B_{10} & \cdots & \nabla_{r_{0L}}B_{10} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\nabla_{r_{10}}B_{K0} & \cdots & \nabla_{r_{K0}}B_{K0} & \nabla_{r_{01}}B_{K0} & \cdots & \nabla_{r_{0L}}B_{K0} \\
\nabla_{r_{10}}B_{01} & \cdots & \nabla_{r_{K0}}B_{01} & \nabla_{r_{01}}B_{01} & \cdots & \nabla_{r_{0L}}B_{01} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\nabla_{r_{10}}B_{0L} & \cdots & \nabla_{r_{K0}}B_{10} & \nabla_{r_{01}}B_{10} & \cdots & \nabla_{r_{0L}}B_{0L}
\end{pmatrix} \\
&= \left(\begin{matrix}
\lambda\frac{p_1}{r_{10}}e_{0|1}\left(\sum_{n=1}^{L}e_{n|1}\right) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda\frac{p_K}{r_{K0}}e_{0|K}\left(\sum_{n=1}^{L}e_{n|K}\right) \\
-(1-\lambda)\frac{q_l}{r_{10}}g_{0|1}g_{1|1} & \cdots & -(1-\lambda)\frac{q_1}{r_{K0}}g_{0|1}g_{K|1} \\
\vdots & \ddots & \vdots \\
-(1-\lambda)\frac{q_L}{r_{10}}g_{0|L}g_{1|L} & \cdots & -(1-\lambda)\frac{q_L}{r_{K0}}g_{0|L}g_{K|L}
\end{matrix}\right. \\
&\qquad \left.\begin{matrix}
-\lambda\frac{p_1}{r_{01}}e_{0|1}e_{1|1} & \cdots & -\lambda\frac{p_1}{r_{0L}}e_{0|1}e_{L|1} \\
\vdots & \ddots & \vdots \\
-\lambda\frac{p_K}{r_{01}}e_{0|K}e_{1|K} & \cdots & -\lambda\frac{p_K}{r_{0L}}e_{0|K}e_{L|K} \\
(1-\lambda)\frac{q_1}{r_{01}}g_{0|1}\left(\sum_{m=1}^{K}g_{m|1}\right) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & (1-\lambda)\frac{q_L}{r_{0L}}g_{0|L}\left(\sum_{m=1}^{K}g_{m|L}\right)
\end{matrix}\right),
\end{aligned}
$$

which at $\mathbf{r}_0 = \mathbf{r}_0^{\text{eq}}$ simplifies to

$$\nabla_{\mathbf{r}_0}\mathbf{B}\left(\mathbf{r}_0^{\text{eq}}\right) = \begin{pmatrix} \lambda\left(\sum_{n=1}^{L} e_{n|1}\right) & \cdots & 0 & -\lambda\frac{r_{10}}{r_{01}}e_{1|1} & \cdots & -\lambda\frac{r_{10}}{r_{0L}}e_{L|1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda\left(\sum_{n=1}^{L} e_{n|K}\right) & -\lambda\frac{r_{K0}}{r_{01}}e_{1|K} & \cdots & -\lambda\frac{r_{K0}}{r_{0L}}e_{L|K} \\ -\left(1-\lambda\right)\frac{r_{01}}{r_{10}}g_{1|1} & \cdots & -\left(1-\lambda\right)\frac{r_{01}}{r_{K0}}g_{K|1} & \left(1-\lambda\right)\left(\sum_{m=1}^{K} g_{m|1}\right) & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\left(1-\lambda\right)\frac{r_{0L}}{r_{10}}g_{1|L} & \cdots & -\left(1-\lambda\right)\frac{r_{0L}}{r_{K0}}g_{K|L} & 0 & \cdots & \left(1-\lambda\right)\left(\sum_{m=1}^{K} g_{m|L}\right) \end{pmatrix}.$$

We can derive these matrix components through repeated application of the chain rule. First we consider the derivative of B_{k0} with respect to, respectively r_{k0}, r_{m0} for $m \neq k$, and r_{0l}:

$$\begin{aligned}
\nabla_{r_{k0}}B_{k0} &= -\frac{p_k\sum_{n=1}^{L}\exp\left[\gamma_{kn}+\lambda\ln\left(\frac{r_{0n}}{r_{k0}}\right)\right]\lambda\left(\frac{r_{0n}}{r_{k0}}\right)^{-1}\left(-\frac{r_{0n}}{\left(r_{k0}\right)^2}\right)}{\left(1+\sum_{n=1}^{L}\exp\left[\gamma_{kn}+\lambda\ln\left(\frac{r_{0n}}{r_{k0}}\right)\right]\right)^2} \\
&= \frac{p_k\sum_{n=1}^{L}\exp\left[\gamma_{kn}+\lambda\ln\left(\frac{r_{0n}}{r_{k0}}\right)\right]\frac{\lambda}{r_{k0}}}{\left(1+\sum_{n=1}^{L}\exp\left[\gamma_{kn}+\lambda\ln\left(\frac{r_{0n}}{r_{k0}}\right)\right]\right)^2} \\
&= p_k e_{0|k}\left(\sum_{n=1}^{L} e_{n|k}\frac{\lambda}{r_{k0}}\right) \\
&= \lambda\frac{p_k}{r_{k0}}e_{0|k}\left(\sum_{n=1}^{L} e_{n|k}\right) \\
&= \lambda\frac{p_k}{r_{k0}}e_{0|k}\left(1-e_{0|k}\right) \\
\nabla_{r_{m0}}B_{k0} &= 0 \\
\nabla_{r_{0l}}B_{k0} &= -\frac{p_k\exp\left[\gamma_{kl}+\lambda\ln\left(\frac{r_{0l}}{r_{k0}}\right)\right]\lambda\left(\frac{r_{0l}}{r_{k0}}\right)^{-1}\left(\frac{1}{r_{k0}}\right)}{\left(1+\sum_{n=1}^{L}\exp\left[\gamma_{kn}+\lambda\ln\left(\frac{r_{0n}}{r_{k0}}\right)\right]\right)^2} \\
&= -\lambda\frac{p_k}{r_{0l}}e_{0|k}e_{l|k}
\end{aligned}$$

Second we consider the derivative of B_{0l} with respect to, respectively r_{k0}, r_{0l}, and r_{0n} for $n \neq l$:

$$
\begin{aligned}
\nabla_{r_{k0}} B_{0l} &= -\frac{q_l \exp\left[\gamma_{kl} - (1-\lambda)\ln\left(\frac{r_{0l}}{r_{k0}}\right)\right]\left(-(1-\lambda)\left(\frac{r_{0l}}{r_{k0}}\right)^{-1}\left(-\frac{r_{0l}}{(r_{k0})^2}\right)\right)}{\left(1+\sum_{m=1}^{K}\exp\left[\gamma_{ml} - (1-\lambda)\ln\left(\frac{r_{0l}}{r_{m0}}\right)\right]\right)^2} \\
&= -(1-\lambda)\frac{q_l}{r_{k0}} g_{0|l} g_{k|l} \\
\nabla_{r_{0l}} B_{0l} &= -\frac{q_l \sum_{m=1}^{K}\exp\left[\gamma_{ml} - (1-\lambda)\ln\left(\frac{r_{0l}}{r_{m0}}\right)\right]\left[-(1-\lambda)\left(\frac{r_{0l}}{r_{m0}}\right)^{-1}\frac{1}{r_{m0}}\right]}{\left(1+\sum_{m=1}^{K}\exp\left[\gamma_{ml} - (1-\lambda)\ln\left(\frac{r_{0l}}{r_{m0}}\right)\right]\right)^2} \\
&= (1-\lambda)\frac{q_l}{r_{0l}} g_{0|l}\left(\sum_{m=1}^{K} g_{m|l}\right) \\
&= (1-\lambda)\frac{q_l}{r_{0l}} g_{0|l}\left(1-g_{0|l}\right) \\
\nabla_{r_{0n}} B_{0l} &= 0.
\end{aligned}
$$

Note that at an equilibrium we have the simplifications

$$
\begin{aligned}
\nabla_{r_{k0}} B_{k0}\left(\mathbf{r}_0^{\mathrm{eq}}\right) &= \lambda\left(\sum_{n=1}^{L} e_{n|k}\right) = \lambda\left(1-e_{0|k}\right) \\
\nabla_{r_{0l}} B_{k0}\left(\mathbf{r}_0^{\mathrm{eq}}\right) &= -\lambda\frac{r_{k0}}{r_{0l}} e_{l|k} \\
\nabla_{r_{k0}} B_{0l}\left(\mathbf{r}_0^{\mathrm{eq}}\right) &= -(1-\lambda)\frac{r_{0l}}{r_{k0}} g_{k|l} \\
\nabla_{r_{0l}} B_{0l}\left(\mathbf{r}_0^{\mathrm{eq}}\right) &= (1-\lambda)\left(\sum_{m=1}^{K} g_{m|l}\right) = (1-\lambda)\left(1-g_{0|l}\right).
\end{aligned}
$$

To derive the special form of the Jacobian matrix (16) we begin by rewriting the Jacobian. Recall that Jacobian matrix associated with (10) is given by $J(\mathbf{r}_0) = I_{K+L} - \nabla_{\mathbf{r}_0}\mathbf{B}(\mathbf{r}_0;\mathbf{p},\mathbf{q},\theta)$ where

$$
J(\mathbf{r}_0) = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix},
$$

with

$$
\begin{aligned}
J_{11} &= I_K - \lambda \cdot \operatorname{diag}\left\{\sum_{n=1}^{L} e_{n|1}\left(\mathbf{r}_0\right), \ldots, \sum_{n=1}^{L} e_{n|K}\left(\mathbf{r}_0\right)\right\} \\
&= I_K - \lambda \cdot \operatorname{diag}\left\{\mathbf{p}\right\}^{-1} \operatorname{diag}\left\{\mathbf{R}\iota_L\right\} \\
J_{12} &= \lambda \begin{pmatrix} \frac{r_{10}}{r_{01}} e_{1|1}\left(\mathbf{r}_0\right) & \cdots & \frac{r_{10}}{r_{0L}} e_{L|1}\left(\mathbf{r}_0\right) \\ \vdots & \ddots & \vdots \\ \frac{r_{K0}}{r_{01}} e_{1|K}\left(\mathbf{r}_0\right) & \cdots & \frac{r_{K0}}{r_{0L}} e_{L|K}\left(\mathbf{r}_0\right) \end{pmatrix} \\
&= \lambda \cdot \operatorname{diag}\left\{\mathbf{r}_{.0}\right\} \operatorname{diag}\left\{\mathbf{p}\right\}^{-1} \begin{pmatrix} r_{11} & \cdots & r_{1L} \\ \vdots & \ddots & \vdots \\ r_{K1} & \cdots & r_{KL} \end{pmatrix} \operatorname{diag}\left\{\mathbf{r}_{0.}\right\}^{-1} \\
&= \lambda \cdot \operatorname{diag}\left\{\mathbf{r}_{.0}\right\} \operatorname{diag}\left\{\mathbf{p}\right\}^{-1} \mathbf{R} \operatorname{diag}\left\{\mathbf{r}_{0.}\right\}^{-1} \\
&= \lambda \cdot \operatorname{diag}\left\{\mathbf{p}\right\}^{-1} \operatorname{diag}\left\{\mathbf{r}_{.0}\right\} \mathbf{R} \operatorname{diag}\left\{\mathbf{r}_{0.}\right\}^{-1} \\
J_{21} &= (1-\lambda) \begin{pmatrix} \frac{r_{0l}}{r_{10}} g_{1|1}\left(\mathbf{r}_0\right) & \cdots & \frac{r_{0l}}{r_{K0}} g_{K|1}\left(\mathbf{r}_0\right) \\ \vdots & \ddots & \vdots \\ \frac{r_{0L}}{r_{10}} g_{1|L}\left(\mathbf{r}_0\right) & \cdots & \frac{r_{0L}}{r_{K0}} g_{K|L}\left(\mathbf{r}_0\right) \end{pmatrix} \\
&= (1-\lambda) \cdot \operatorname{diag}\left\{\mathbf{r}_{0.}\right\} \operatorname{diag}\left\{\mathbf{q}\right\}^{-1} \begin{pmatrix} r_{11} & \cdots & r_{1K} \\ \vdots & \ddots & \vdots \\ r_{1L} & \cdots & r_{KL} \end{pmatrix} \operatorname{diag}\left\{\mathbf{r}_{.0}\right\}^{-1} \\
&= (1-\lambda) \cdot \operatorname{diag}\left\{\mathbf{r}_{0.}\right\} \operatorname{diag}\left\{\mathbf{q}\right\}^{-1} \mathbf{R}' \operatorname{diag}\left\{\mathbf{r}_{.0}\right\}^{-1} \\
&= (1-\lambda) \cdot \operatorname{diag}\left\{\mathbf{q}\right\}^{-1} \operatorname{diag}\left\{\mathbf{r}_{0.}\right\} \mathbf{R}' \operatorname{diag}\left\{\mathbf{r}_{.0}\right\}^{-1} \\
J_{22} &= I_L - (1-\lambda) \cdot \operatorname{diag}\left\{\sum_{m=1}^{K} g_{m|1}\left(\mathbf{r}_0\right), \ldots, \sum_{m=1}^{K} g_{m|L}\left(\mathbf{r}_0\right)\right\} \\
&= I_L - (1-\lambda) \cdot \operatorname{diag}\left\{\mathbf{q}\right\}^{-1} \operatorname{diag}\left\{\mathbf{R}'\iota_K\right\}.
\end{aligned}
$$

Tedious manipulation then gives

$$
\begin{aligned}
J(\mathbf{r}_0) &= \begin{pmatrix} I_K - \lambda \cdot \mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{diag}\{\mathbf{R}\iota_L\} & 0 \\ 0 & I_L - (1-\lambda)\cdot \mathrm{diag}\{\mathbf{q}\}^{-1}\mathrm{diag}\{\mathbf{R}'\iota_K\} \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & \lambda \cdot \mathrm{diag}\{\mathbf{p}\}^{-1}\mathrm{diag}\{\mathbf{r}_{.0}\}\mathbf{R}\mathrm{diag}\{\mathbf{r}_{0.}\}^{-1} \\ (1-\lambda)\cdot \mathrm{diag}\{\mathbf{q}\}^{-1}\mathrm{diag}\{\mathbf{r}_{0.}\}\mathbf{R}'\mathrm{diag}\{\mathbf{r}_{.0}\}^{-1} & 0 \end{pmatrix} \\
&= \begin{pmatrix} I_K & 0 \\ 0 & I_L \end{pmatrix} - \begin{pmatrix} \lambda \cdot \mathrm{diag}\{\mathbf{p}\}^{-1} & 0 \\ 0 & (1-\lambda)\cdot \mathrm{diag}\{\mathbf{q}\}^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{diag}\{\mathbf{R}\iota_L\} & 0 \\ 0 & \mathrm{diag}\{\mathbf{R}'\iota_K\} \end{pmatrix} \\
&\quad + \begin{pmatrix} \lambda \cdot \mathrm{diag}\{\mathbf{p}\}^{-1} & 0 \\ 0 & (1-\lambda)\cdot \mathrm{diag}\{\mathbf{q}\}^{-1} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{diag}\{\mathbf{r}_{.0}\}\mathbf{R}\mathrm{diag}\{\mathbf{r}_{0.}\}^{-1} \\ \mathrm{diag}\{\mathbf{r}_{0.}\}\mathbf{R}'\mathrm{diag}\{\mathbf{r}_{.0}\}^{-1} & 0 \end{pmatrix} \\
&= \begin{pmatrix} I_K & 0 \\ 0 & I_L \end{pmatrix} - \begin{pmatrix} \lambda \cdot \mathrm{diag}\{\mathbf{p}\}^{-1} & 0 \\ 0 & (1-\lambda)\cdot \mathrm{diag}\{\mathbf{q}\}^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{diag}\{\mathbf{R}\iota_L\} & 0 \\ 0 & \mathrm{diag}\{\mathbf{R}'\iota_K\} \end{pmatrix} \\
&\quad + \begin{pmatrix} \lambda \cdot \mathrm{diag}\{\mathbf{p}\}^{-1} & 0 \\ 0 & (1-\lambda)\cdot \mathrm{diag}\{\mathbf{q}\}^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{diag}\{\mathbf{r}_{.0}\} & 0 \\ 0 & \mathrm{diag}\{\mathbf{r}_{0.}\} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{R} \\ \mathbf{R}' & 0 \end{pmatrix} \\
&\quad \begin{pmatrix} \mathrm{diag}\{\mathbf{r}_{.0}\}^{-1} & 0 \\ 0 & \mathrm{diag}\{\mathbf{r}_{0.}\}^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \lambda \cdot \mathrm{diag}\{\mathbf{p}\}^{-1} & 0 \\ 0 & (1-\lambda)\cdot \mathrm{diag}\{\mathbf{q}\}^{-1} \end{pmatrix} \left[\begin{pmatrix} \frac{\mathrm{diag}\{\mathbf{p}\}}{\lambda} & 0 \\ 0 & \frac{\mathrm{diag}\{\mathbf{q}\}}{1-\lambda} \end{pmatrix} - \begin{pmatrix} \mathrm{diag}\{\mathbf{R}\iota_L\} & 0 \\ 0 & \mathrm{diag}\{\mathbf{R}'\iota_K\} \end{pmatrix} \right. \\
&\quad \left. + \begin{pmatrix} \mathrm{diag}\{\mathbf{r}_{.0}\} & 0 \\ 0 & \mathrm{diag}\{\mathbf{r}_{0.}\} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{R} \\ \mathbf{R}' & 0 \end{pmatrix} \begin{pmatrix} \mathrm{diag}\{\mathbf{r}_{.0}\}^{-1} & 0 \\ 0 & \mathrm{diag}\{\mathbf{r}_{0.}\}^{-1} \end{pmatrix} \right] \\
&= \begin{pmatrix} \lambda \cdot \mathrm{diag}\{\mathbf{p}\}^{-1} & 0 \\ 0 & (1-\lambda)\cdot \mathrm{diag}\{\mathbf{q}\}^{-1} \end{pmatrix} \left[\begin{pmatrix} \frac{\mathrm{diag}\{\mathbf{p}-\lambda\mathbf{R}\iota_L\}}{\lambda} & 0 \\ 0 & \frac{\mathrm{diag}\{\mathbf{q}-(1-\lambda)\mathbf{R}'\iota_K\}}{1-\lambda} \end{pmatrix} \right. \\
&\quad \left. + \begin{pmatrix} \mathrm{diag}\{\mathbf{r}_{.0}\} & 0 \\ 0 & \mathrm{diag}\{\mathbf{r}_{0.}\} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{R} \\ \mathbf{R}' & 0 \end{pmatrix} \begin{pmatrix} \mathrm{diag}\{\mathbf{r}_{.0}\}^{-1} & 0 \\ 0 & \mathrm{diag}\{\mathbf{r}_{0.}\}^{-1} \end{pmatrix} \right] \\
&= \begin{pmatrix} \mathrm{diag}\{\mathbf{p}\}^{-1} & 0 \\ 0 & \mathrm{diag}\{\mathbf{q}\}^{-1} \end{pmatrix} \left[\begin{pmatrix} \mathrm{diag}\{\mathbf{p}-\lambda\mathbf{R}\iota_L\} & 0 \\ 0 & \mathrm{diag}\{\mathbf{q}-(1-\lambda)\mathbf{R}'\iota_K\} \end{pmatrix} \right. \\
&\quad \left. + \begin{pmatrix} \mathrm{diag}\{\mathbf{r}_{.0}\} & 0 \\ 0 & \mathrm{diag}\{\mathbf{r}_{0.}\} \end{pmatrix} \begin{pmatrix} 0 & \lambda\mathbf{R} \\ (1-\lambda)\mathbf{R}' & 0 \end{pmatrix} \begin{pmatrix} \mathrm{diag}\{\mathbf{r}_{.0}\}^{-1} & 0 \\ 0 & \mathrm{diag}\{\mathbf{r}_{0.}\}^{-1} \end{pmatrix} \right],
\end{aligned}
$$

which is of the form $C(\mathbf{r}_0)^{-1}\left(A(\mathbf{r}_0) + U(\mathbf{r}_0)B(\mathbf{r}_0)U(\mathbf{r}_0)^{-1}\right)$ as defined.

Evaluating $H_{11}^{-1}H_{12}H_{22}^{-1}H_{21}H_{11}^{-1}$ yields

$$\begin{aligned}
H_{11}^{-1}H_{12}H_{22}^{-1}H_{21}H_{11}^{-1} \quad = \quad & \operatorname{diag}\left\{\frac{p_1}{(1-\lambda)\,p_1+\lambda r_{10}},\ldots,\frac{p_K}{(1-\lambda)\,p_K+\lambda r_{K0}}\right\} \\
& \times\begin{pmatrix} \lambda\frac{r_{11}}{p_1} & \cdots & \lambda\frac{r_{1L}}{p_1} \\ \vdots & \ddots & \vdots \\ \lambda\frac{r_{K1}}{p_K} & \cdots & \lambda\frac{r_{KL}}{p_K} \end{pmatrix} \\
& \times\operatorname{diag}\left\{\frac{q_1}{\lambda q_1+(1-\lambda)\,r_{01}},\ldots,\frac{q_L}{\lambda q_L+(1-\lambda)\,r_{0L}}\right\} \\
& \times\begin{pmatrix} \frac{(1-\lambda)r_{11}}{q_1} & \cdots & \frac{(1-\lambda)r_{K1}}{q_1} \\ \vdots & \ddots & \vdots \\ \frac{(1-\lambda)r_{1L}}{q_L} & \cdots & \frac{(1-\lambda)r_{KL}}{q_L} \end{pmatrix} \\
& \times\operatorname{diag}\left\{\frac{p_1}{(1-\lambda)\,p_1+\lambda r_{10}},\ldots,\frac{p_K}{(1-\lambda)\,p_K+\lambda r_{K0}}\right\} \\
= \quad & \lambda\begin{pmatrix} \frac{1}{(1-\lambda)p_1+\lambda r_{10}}\frac{q_1}{\lambda q_1+(1-\lambda)r_{01}}r_{11} & \cdots & \frac{1}{(1-\lambda)p_1+\lambda r_{10}}\frac{q_L}{\lambda q_L+(1-\lambda)r_{0L}}r_{1L} \\ \vdots & \ddots & \vdots \\ \frac{1}{(1-\lambda)p_K+\lambda r_{K0}}\frac{q_1}{\lambda q_1+(1-\lambda)r_{01}}r_{K1} & \cdots & \frac{1}{(1-\lambda)p_K+\lambda r_{K0}}\frac{q_L}{\lambda q_L+(1-\lambda)r_{0L}}r_{KL} \end{pmatrix} \\
& \times\begin{pmatrix} \frac{(1-\lambda)r_{11}}{q_1} & \cdots & \frac{(1-\lambda)r_{K1}}{q_1} \\ \vdots & \ddots & \vdots \\ \frac{(1-\lambda)r_{1L}}{q_L} & \cdots & \frac{(1-\lambda)r_{KL}}{q_L} \end{pmatrix} \\
& \times\operatorname{diag}\left\{\frac{p_1}{(1-\lambda)\,p_1+\lambda r_{10}},\ldots,\frac{p_K}{(1-\lambda)\,p_K+\lambda r_{K0}}\right\} \\
= \lambda\,(1-\lambda) \quad & \begin{pmatrix} \frac{1}{(1-\lambda)p_1+\lambda r_{10}}\sum_{n=1}^{L}\frac{r_{1n}r_{1n}}{\lambda q_n+(1-\lambda)r_{0n}} & \cdots & \frac{1}{(1-\lambda)p_1+\lambda r_{10}}\sum_{n=1}^{L}\frac{r_{1n}r_{Kn}}{\lambda q_n+(1-\lambda)r_{0n}} \\ \vdots & \ddots & \vdots \\ \frac{1}{(1-\lambda)p_K+\lambda r_{K0}}\sum_{n=1}^{L}\frac{r_{Kn}r_{1n}}{\lambda q_n+(1-\lambda)r_{0n}} & \cdots & \frac{1}{(1-\lambda)p_K+\lambda r_{K0}}\sum_{n=1}^{L}\frac{r_{Kn}r_{Kn}}{\lambda q_n+(1-\lambda)r_{0n}} \end{pmatrix} \\
& \times\operatorname{diag}\left\{\frac{p_1}{(1-\lambda)\,p_1+\lambda r_{10}},\ldots,\frac{p_K}{(1-\lambda)\,p_K+\lambda r_{K0}}\right\} \\
= \quad & \lambda\,(1-\lambda)\begin{pmatrix} \frac{1}{(1-\lambda)p_1+\lambda r_{10}}\frac{p_1}{(1-\lambda)p_1+\lambda r_{10}}\sum_{n=1}^{L}\frac{r_{1n}r_{1n}}{\lambda q_n+(1-\lambda)r_{0n}} & \cdots & \frac{1}{(1-\lambda)p_1+\lambda r_{10}}\frac{p_K}{(1-\lambda)p_K+\lambda r_{K0}}\sum_{n=1}^{L}\frac{r_{1n}r_{Kn}}{\lambda q_n+(1-\lambda)r_{0n}} \\ \vdots & \ddots & \vdots \\ \frac{1}{(1-\lambda)p_K+\lambda r_{K0}}\frac{p_1}{(1-\lambda)p_1+\lambda r_{10}}\sum_{n=1}^{L}\frac{r_{Kn}r_{1n}}{\lambda q_n+(1-\lambda)r_{0n}} & \cdots & \frac{1}{(1-\lambda)p_K+\lambda r_{K0}}\frac{p_K}{(1-\lambda)p_K+\lambda r_{K0}}\sum_{n=1}^{L}\frac{r_{Kn}r_{Kn}}{\lambda q_n+(1-\lambda)r_{0n}} \end{pmatrix}.
\end{aligned}$$

Evaluating $H_{22}^{-1}H_{21}H_{11}^{-1}H_{12}H_{22}^{-1}$ yields

$$\begin{aligned}
H_{22}^{-1}H_{21}H_{11}^{-1}H_{12}H_{22}^{-1} &= \operatorname{diag}\left\{\frac{q_1}{\lambda q_1+(1-\lambda)\,r_{01}},\dots,\frac{q_L}{\lambda q_L+(1-\lambda)\,r_{0L}}\right\} \\
&\quad\times\begin{pmatrix} \frac{(1-\lambda)r_{11}}{q_1} & \cdots & \frac{(1-\lambda)r_{K1}}{q_1} \\ \vdots & \ddots & \vdots \\ \frac{(1-\lambda)r_{1L}}{q_L} & \cdots & \frac{(1-\lambda)r_{KL}}{q_L} \end{pmatrix} \\
&\quad\times\operatorname{diag}\left\{\frac{p_1}{(1-\lambda)\,p_1+\lambda r_{10}},\dots,\frac{p_K}{(1-\lambda)\,p_K+\lambda r_{K0}}\right\} \\
&\quad\times\begin{pmatrix} \lambda\frac{r_{11}}{p_1} & \cdots & \lambda\frac{r_{1L}}{p_1} \\ \vdots & \ddots & \vdots \\ \lambda\frac{r_{K1}}{p_K} & \cdots & \lambda\frac{r_{KL}}{p_K} \end{pmatrix} \\
&\quad\times\operatorname{diag}\left\{\frac{q_1}{\lambda q_1+(1-\lambda)\,r_{01}},\dots,\frac{q_L}{\lambda q_L+(1-\lambda)\,r_{0L}}\right\} \\
&= (1-\lambda)\begin{pmatrix} \frac{1}{\lambda q_1+(1-\lambda)r_{01}}\frac{p_1}{(1-\lambda)p_1+\lambda r_{10}}r_{11} & \cdots & \frac{1}{\lambda q_1+(1-\lambda)r_{01}}\frac{p_K}{(1-\lambda)p_K+\lambda r_{K0}}r_{K1} \\ \vdots & \ddots & \vdots \\ \frac{1}{\lambda q_L+(1-\lambda)r_{0L}}\frac{p_1}{(1-\lambda)p_1+\lambda r_{10}}r_{1L} & \cdots & \frac{1}{\lambda q_L+(1-\lambda)r_{0L}}\frac{p_K}{(1-\lambda)p_K+\lambda r_{K0}}r_{KL} \end{pmatrix} \\
&\quad\times\begin{pmatrix} \lambda\frac{r_{11}}{p_1} & \cdots & \lambda\frac{r_{1L}}{p_1} \\ \vdots & \ddots & \vdots \\ \lambda\frac{r_{K1}}{p_K} & \cdots & \lambda\frac{r_{KL}}{p_K} \end{pmatrix}\times\operatorname{diag}\left\{\frac{q_1}{\lambda q_1+(1-\lambda)\,r_{01}},\dots,\frac{q_L}{\lambda q_L+(1-\lambda)\,r_{0L}}\right\} \\
&= \lambda\,(1-\lambda) \\
&\quad\times\begin{pmatrix} \frac{1}{\lambda q_1+(1-\lambda)r_{01}}\sum_{m=1}^{K}\frac{r_{m1}r_{m1}}{(1-\lambda)p_m+\lambda r_{m0}} & \cdots & \frac{1}{\lambda q_1+(1-\lambda)r_{01}}\sum_{m=1}^{K}\frac{r_{m1}r_{mL}}{(1-\lambda)p_m+\lambda r_{m0}} \\ \vdots & \ddots & \vdots \\ \frac{1}{\lambda q_L+(1-\lambda)r_{0L}}\sum_{m=1}^{K}\frac{r_{mL}r_{m1}}{(1-\lambda)p_m+\lambda r_{m0}} & \cdots & \frac{1}{\lambda q_L+(1-\lambda)r_{0L}}\sum_{m=1}^{K}\frac{r_{mL}r_{mL}}{(1-\lambda)p_m+\lambda r_{m0}} \end{pmatrix} \\
&\quad\times\operatorname{diag}\left\{\frac{q_1}{\lambda q_1+(1-\lambda)\,r_{01}},\dots,\frac{q_L}{\lambda q_L+(1-\lambda)\,r_{0L}}\right\} \\
&= \lambda\,(1-\lambda)\begin{pmatrix} \frac{1}{\lambda q_1+(1-\lambda)r_{01}}\frac{q_1}{\lambda q_1+(1-\lambda)r_{01}}\sum_{m=1}^{K}\frac{r_{m1}r_{m1}}{(1-\lambda)p_m+\lambda r_{m0}} & \cdots & \frac{1}{\lambda q_1+(1-\lambda)r_{01}}\frac{q_L}{\lambda q_L+(1-\lambda)r_{0L}}\sum_{m=1}^{K}\frac{r_{m1}r_{mL}}{(1-\lambda)p_m+\lambda r_{m0}} \\ \vdots & \ddots & \vdots \\ \frac{1}{\lambda q_L+(1-\lambda)r_{0L}}\frac{q_1}{\lambda q_1+(1-\lambda)r_{01}}\sum_{m=1}^{K}\frac{r_{mL}r_{m1}}{(1-\lambda)p_m+\lambda r_{m0}} & \cdots & \frac{1}{\lambda q_L+(1-\lambda)r_{0L}}\frac{q_L}{\lambda q_L+(1-\lambda)r_{0L}}\sum_{m=1}^{K}\frac{r_{mL}r_{mL}}{(1-\lambda)p_m+\lambda r_{m0}} \end{pmatrix}.
\end{aligned}$$

To derive the form of $\frac{\partial \mathbf{B}}{\partial \gamma_{kl}}$ stated in the text of the proof of Theorem 1 we differentiate

$$\frac{\partial \mathbf{B}}{\partial \gamma_{kl}} = \begin{pmatrix} 0 \\ \vdots \\ -r_{k0}\left(\frac{\exp\left(\gamma_{kl}+\lambda\ln\left(\frac{r_{0l}}{r_{k0}}\right)\right)}{1+\sum_{n=1}^{L}\exp\left(\gamma_{kn}+\lambda\ln\left(\frac{r_{0n}}{r_{k0}}\right)\right)}\right) \\ \vdots \\ 0 \\ \vdots \\ -r_{0l}\left(\frac{\exp\left(\gamma_{kl}-(1-\lambda)\ln\left(\frac{r_{0l}}{r_{k0}}\right)\right)}{1+\sum_{m=1}^{K}\exp\left(\gamma_{ml}-(1-\lambda)\ln\left(\frac{r_{0l}}{r_{m0}}\right)\right)}\right) \\ \vdots \\ 0 \end{pmatrix},$$

and observe that $\exp\left(\gamma_{kl}+\lambda\ln\left(\frac{r_{0l}}{r_{k0}}\right)\right) = \exp(\gamma_{kl})\left(\frac{r_{0l}}{r_{k0}}\right)^{\lambda}$ and $\exp\left(\gamma_{kl}-(1-\lambda)\ln\left(\frac{r_{0l}}{r_{k0}}\right)\right) = \exp(\gamma_{kl})\left(\frac{r_{k0}}{r_{0l}}\right)^{1-\lambda}$ and hence that

$$\frac{\exp\left(\gamma_{kl}+\lambda\ln\left(\frac{r_{0l}}{r_{k0}}\right)\right)}{1+\sum_{n=1}^{L}\exp\left(\gamma_{kn}+\lambda\ln\left(\frac{r_{0n}}{r_{k0}}\right)\right)} = \exp(\gamma_{kl})\left(\frac{r_{0l}}{r_{k0}}\right)^{\lambda}\frac{r_{k0}}{p_k} = \frac{\exp(\gamma_{kl})\, r_{k0}^{1-\lambda} r_{0l}^{\lambda}}{p_k} = \frac{r_{kl}}{p_k}$$

$$\frac{\exp\left(\gamma_{kl}-(1-\lambda)\ln\left(\frac{r_{0l}}{r_{k0}}\right)\right)}{1+\sum_{m=1}^{K}\exp\left(\gamma_{ml}-(1-\lambda)\ln\left(\frac{r_{0l}}{r_{m0}}\right)\right)} = \exp(\gamma_{kl})\left(\frac{r_{k0}}{r_{0l}}\right)^{1-\lambda}\frac{r_{0l}}{q_l} = \frac{\exp(\gamma_{kl})\, r_{k0}^{1-\lambda} r_{0l}^{\lambda}}{q_l} = \frac{r_{kl}}{q_l}.$$