Detailed calculations for an "An econometric model of link formation with degree heterogeneity"

This document contains calculation details underlying some of the results presented in the paper "An econometric model of link formation with degree heterogeneity" and/or its associated supplemental web appendices. It is not meant for formal publication. All notation is as defined in the main text unless explicitly noted otherwise. Equation numbering continues in sequence with that established in the main text and supplemental web appendices.

Calculation details for Appendix A.1

To derive equation (25) of Appendix A I evaluate

$$\begin{aligned}\mathbb{E}\left[l_{ij,kl}(\beta)\right] &= \mathbb{E}\left[\left|S_{ij,kl}\right|\left\{S_{ij,kl}\tilde{W}_{ij,kl}'\beta-\ln\left[1+\exp\left(S_{ij,kl}\tilde{W}_{ij,kl}'\beta\right)\right]\right\}\right]\\ &= \Pr\left(S_{ij,kl}\in\{-1,1\}\right)\mathbb{E}\left[S_{ij,kl}\tilde{W}_{ij,kl}'\beta-\ln\left[1+\exp\left(S_{ij,kl}\tilde{W}_{ij,kl}'\beta\right)\right]\middle| S_{ij,kl}\in\{-1,1\}\right].\end{aligned}$$

The law-of-iterated expectations applied to the second term in the product above then gives, evaluating the inner expectation only,

$$\begin{aligned}&\mathbb{E}\left[S_{ij,kl}\tilde{W}_{ij,kl}'\beta-\ln\left[1+\exp\left(S_{ij,kl}\tilde{W}_{ij,kl}'\beta\right)\right]\middle|\mathbf{X},S_{ij,kl}\in\{-1,1\}\right]\\ &=\left\{\tilde{W}_{ij,kl}'\beta-\ln\left[1+\exp\left(\tilde{W}_{ij,kl}'\beta\right)\right]\right\}q_{ij,kl}\\ &\quad+\left\{-\tilde{W}_{ij,kl}'\beta-\ln\left[1+\exp\left(-\tilde{W}_{ij,kl}'\beta\right)\right]\right\}(1-q_{ij,kl})\\ &=\ln\left(q_{ij,kl}(\beta)\right)q_{ij,kl}+\ln\left(1-q_{ij,kl}(\beta)\right)(1-q_{ij,kl})\\ &=-\left\{q_{ij,kl}\ln\left(\frac{q_{ij,kl}}{q_{ij,kl}(\beta)}\right)+(1-q_{ij,kl})\ln\left(\frac{1-q_{ij,kl}}{1-q_{ij,kl}(\beta)}\right)\right\}\\ &\quad+\left\{q_{ij,kl}\ln\left(q_{ij,kl}\right)+(1-q_{ij,kl})\ln\left(1-q_{ij,kl}\right)\right\}\\ &\quad-\left\{D_{\mathrm{KL}}\left(q_{ij,kl}\|\,q_{ij,kl}(\beta)\right)-\mathbf{S}\left(q_{ij,kl}\right)\right\},\end{aligned}$$

as needed.

Calculation details for Appendix A.2

To double check that (34) is the desired projection we can verify the orthogonality condition

$$
\begin{aligned}
\mathbb{E}\left[\left(U_N-U_N^*\right)\sum_{i<j} g_{ij}\left(X_i,X_j,A_i,A_j,U_{ij}\right)\right] &= \sum_{i<j}\mathbb{E}\left[\left(U_N-U_N^*\right) g_{ij}\left(X_i,X_j,A_i,A_j,U_{ij}\right)\right] \\
&= \sum_{i<j}\mathbb{E}\left[\mathbb{E}\left[\left. U_N-U_N^*\right| X_i,X_j,A_i,A_j,U_{ij}\right] g_{ij}\left(X_i,X_j,A_i,A_j,U_{ij}\right)\right] \\
&= 0.
\end{aligned}
$$

for $g_{ij}(\cdot)$ an arbitrary function of $(X_i, X_j, A_i, A_j, U_{ij})$.

To show that $\bar{s}_{ij}(\beta_0)$ and $\bar{s}_{kl}(\beta_0)$ are uncorrelated for $\{i,j\} \neq \{k,l\}$ use iterated expectations to evaluate:

$$
\begin{aligned}
\mathbb{E}\left[\bar{s}_{ij}(\beta_0)\,\bar{s}_{kl}(\beta_0)'\right] &= \mathbb{E}\left[\mathbb{E}\left[\left. s_{ijmp}(\beta_0)\right| X_i,X_j,A_i,A_j,U_{ij}\right]\mathbb{E}\left[\left. s_{klm'p'}(\beta_0)\right| X_k,X_l,A_k,A_l,U_{kl}\right]'\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\left. s_{ijmp}(\beta_0)\right| X_i,X_j,A_i,A_j,U_{ij}\right]\right.\right. \\
&\quad \left.\left.\times\mathbb{E}\left[\left. s_{klm'p'}(\beta_0)\right| X_k,X_l,A_k,A_l,U_{kl}\right]'\right|\mathbf{X},\mathbf{A}\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left.\mathbb{E}\left[\left. s_{ijmp}(\beta_0)\right| X_i,X_j,A_i,A_j,U_{ij}\right]\right|\mathbf{X},\mathbf{A}\right]\right. \\
&\quad \left.\times\,\mathbb{E}\left[\left.\mathbb{E}\left[\left. s_{klm'p'}(\beta_0)\right| X_k,X_l,A_k,A_l,U_{kl}\right]\right|\mathbf{X},\mathbf{A}\right]'\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left. s_{ijmp}(\beta_0)\right|\mathbf{X},\mathbf{A}\right]\mathbb{E}\left[\left. s_{klm'p'}(\beta_0)\right|\mathbf{X},\mathbf{A}\right]'\right] \\
&= 0.
\end{aligned}
$$

A derivation of (40) is:

$$
\begin{aligned}
\mathbb{E}\left[\left(R_{\mathbf{i}}-Y_{\mathbf{i}}\right)\frac{1}{\sqrt{n}}f'\left(\frac{1}{\sqrt{n}}\left(\mathbf{Z}_{\mathbf{i}}^0\right)'\iota\right)\right] &= \mathbb{E}\left[\left(R_{\mathbf{i}}-Y_{\mathbf{i}}\right)\frac{1}{\sqrt{n}}f'\left(\frac{1}{\sqrt{n}}\left[\sum_{\mathbf{j}=1}^{\mathbf{i}-1}R_{\mathbf{j}}+\sum_{\mathbf{j}=\mathbf{i}+1}^{\mathbf{n}}Y_{\mathbf{j}}\right]\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left.\left(R_{\mathbf{i}}-Y_{\mathbf{i}}\right)\frac{1}{\sqrt{n}}f'\left(\frac{1}{\sqrt{n}}\left[\sum_{\mathbf{j}=1}^{\mathbf{i}-1}R_{\mathbf{j}}+\sum_{\mathbf{j}=\mathbf{i}+1}^{\mathbf{n}}Y_{\mathbf{j}}\right]\right)\right|\mathbf{X},\mathbf{A}\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left. R_{\mathbf{i}}-Y_{\mathbf{i}}\right|\mathbf{X},\mathbf{A}\right]\mathbb{E}\left[\left.\frac{1}{\sqrt{n}}f'\left(\frac{1}{\sqrt{n}}\left[\sum_{\mathbf{j}=1}^{\mathbf{i}-1}R_{\mathbf{j}}+\sum_{\mathbf{j}=\mathbf{i}+1}^{\mathbf{n}}Y_{\mathbf{j}}\right]\right)\right|\mathbf{X},\mathbf{A}\right]\right] \\
&= \mathbb{E}\left[0\cdot\mathbb{E}\left[\left.\frac{1}{\sqrt{n}}f'\left(\frac{1}{\sqrt{n}}\left[\sum_{\mathbf{j}=1}^{\mathbf{i}-1}R_{\mathbf{j}}+\sum_{\mathbf{j}=\mathbf{i}+1}^{\mathbf{n}}Y_{\mathbf{j}}\right]\right)\right|\mathbf{X},\mathbf{A}\right]\right] \\
&= 0.
\end{aligned}
$$

A derivation of (41) is:

$$
\begin{aligned}
\mathbb{E}\left[\left(R_{\mathbf{i}}^2 - Y_{\mathbf{i}}^2\right)\frac{1}{n}f''\left(\frac{1}{\sqrt{n}}\left(\mathbf{Z}_{\mathbf{i}}^0\right)'\iota\right)\right] &= \mathbb{E}\left[\left(R_{\mathbf{i}}^2 - Y_{\mathbf{i}}^2\right)\frac{1}{n}f''\left(\frac{1}{\sqrt{n}}\left[\sum_{\mathbf{j}=1}^{\mathbf{i-1}} R_{\mathbf{j}} + \sum_{\mathbf{j}=\mathbf{i}+1}^{\mathbf{n}} Y_{\mathbf{j}}\right]\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(R_{\mathbf{i}}^2 - Y_{\mathbf{i}}^2\right)\frac{1}{n}f''\left(\frac{1}{\sqrt{n}}\left[\sum_{\mathbf{j}=1}^{\mathbf{i-1}} R_{\mathbf{j}} + \sum_{\mathbf{j}=\mathbf{i}+1}^{\mathbf{n}} Y_{\mathbf{j}}\right]\right)\middle|\,\mathbf{X},\mathbf{A}\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[R_{\mathbf{i}}^2 - Y_{\mathbf{i}}^2\middle|\,\mathbf{X},\mathbf{A}\right]\mathbb{E}\left[\frac{1}{n}f''\left(\frac{1}{\sqrt{n}}\left[\sum_{\mathbf{j}=1}^{\mathbf{i-1}} R_{\mathbf{j}} + \sum_{\mathbf{j}=\mathbf{i}+1}^{\mathbf{n}} Y_{\mathbf{j}}\right]\right)\middle|\,\mathbf{X},\mathbf{A}\right]\right] \\
&= \mathbb{E}\left[0\cdot\mathbb{E}\left[\frac{1}{n}f''\left(\frac{1}{\sqrt{n}}\left[\sum_{\mathbf{j}=1}^{\mathbf{i-1}} R_{\mathbf{j}} + \sum_{\mathbf{j}=\mathbf{i}+1}^{\mathbf{n}} Y_{\mathbf{j}}\right]\right)\middle|\,\mathbf{X},\mathbf{A}\right]\right] \\
&= 0.
\end{aligned}
$$

Calculation details for proof of Lemma 4

To derive (42), as given in the proof of Lemma 4, observe that tedious calculation gives

$$
\begin{aligned}
\nabla\varphi_{\mathbf{A}}\left(\mathbf{A}\right) &= \begin{pmatrix}
\frac{\sum_{j\neq 1}\frac{\exp\left(W'_{1j}\beta\right)\exp\left(W'_{1j}\beta+A_1(\beta)\right)}{\left[\exp\left(-A_j(\beta)\right)+\exp\left(W'_{1j}\beta+A_1(\beta)\right)\right]^2}}{\sum_{j\neq 1}\frac{\exp\left(W'_{1j}\beta\right)}{\exp\left(-A_j(\beta)\right)+\exp\left(W'_{1j}\beta+A_1(\beta)\right)}} & -\frac{\frac{\exp\left(W'_{12}\beta\right)\exp(-A_2(\beta))}{\left[\exp(-A_2(\beta))+\exp\left(W'_{12}\beta+A_1(\beta)\right)\right]^2}}{\sum_{j\neq 1}\frac{\exp\left(W'_{1j}\beta\right)}{\exp\left(-A_j(\beta)\right)+\exp\left(W'_{1j}\beta+A_1(\beta)\right)}} & \cdots & -\frac{\frac{\exp\left(W'_{1N}\beta\right)\exp(-A_N(\beta))}{\left[\exp(-A_N(\beta))+\exp\left(W'_{1N}\beta+A_1(\beta)\right)\right]^2}}{\sum_{j\neq 1}\frac{\exp\left(W'_{1j}\beta\right)}{\exp\left(-A_j(\beta)\right)+\exp\left(Z'_{1j}\beta+A_1(\beta)\right)}} \\
-\frac{\frac{\exp\left(W'_{12}\beta\right)\exp(-A_1(\beta))}{\left[\exp(-A_1(\beta))+\exp\left(W'_{12}\beta+A_2(\beta)\right)\right]^2}}{\sum_{j\neq 2}\frac{\exp\left(W'_{2j}\beta\right)}{\exp\left(-A_j(\beta)\right)+\exp\left(W'_{2j}\beta+A_2(\beta)\right)}} & \frac{\sum_{j\neq 2}\frac{\exp\left(W'_{2j}\beta\right)\exp\left(W'_{2j}\beta+A_2(\beta)\right)}{\left[\exp\left(A_j(\beta)\right)+\exp\left(W'_{2j}\beta+A_2(\beta)\right)\right]^2}}{\sum_{j\neq 2}\frac{\exp\left(W'_{2j}\beta\right)}{\exp\left(-A_j(\beta)\right)+\exp\left(W'_{2j}\beta+A_2(\beta)\right)}} & \cdots & -\frac{\frac{\exp\left(W'_{2N}\beta\right)\exp(-A_N(\beta))}{\left[\exp(-A_N(\beta))+\exp\left(W'_{2N}\beta+A_2(\beta)\right)\right]^2}}{\sum_{j\neq 2}\frac{\exp\left(W'_{2j}\beta\right)}{\exp\left(-A_j(\beta)\right)+\exp\left(W'_{2j}\beta+A_2(\beta)\right)}} \\
\vdots & & \ddots & \vdots \\
-\frac{\frac{\exp\left(W'_{1N}\beta\right)\exp(-A_1(\beta))}{\left[\exp(-A_1(\beta))+\exp\left(W'_{1N}\beta+A_N(\beta)\right)\right]^2}}{\sum_{j\neq N}\frac{\exp\left(W'_{Nj}\beta\right)}{\exp\left(-A_j(\beta)\right)+\exp\left(W'_{Nj}\beta+A_N(\beta)\right)}} & \cdots & \cdots & \frac{\sum_{j\neq N}\frac{\exp\left(W'_{Nj}\beta\right)\exp\left(W'_{Nj}\beta+A_N(\beta)\right)}{\left[\exp\left(A_j(\beta)\right)+\exp\left(W'_{Nj}\beta+A_N(\beta)\right)\right]^2}}{\sum_{j\neq N}\frac{\exp\left(Z'_{Nj}\beta\right)}{\exp\left(-A_j(\beta)\right)+\exp\left(Z'_{Nj}\beta+A_N(\beta)\right)}}
\end{pmatrix} \\
&= \begin{pmatrix}
\frac{\sum_{j\neq 1} r_{1j}p_{1j}}{\sum_{j\neq 1} r_{1j}} & -\frac{r_{12}(1-p_{12})}{\sum_{j\neq 1} r_{1j}} & \cdots & -\frac{r_{1N}(1-p_{1N})}{\sum_{j\neq 1} r_{1j}} \\
-\frac{r_{21}(1-p_{12})}{\sum_{j\neq 2} r_{2j}} & \frac{\sum_{j\neq 2} r_{2j}p_{2j}}{\sum_{j\neq 2} r_{2j}} & \cdots & -\frac{r_{2N}(1-p_{2N})}{\sum_{j\neq 2} r_{2j}} \\
\vdots & & \ddots & \vdots \\
-\frac{r_{N1}(1-p_{1N})}{\sum_{j\neq N} r_{Nj}} & -\frac{r_{2N}(1-p_{2N})}{\sum_{j\neq N} r_{Nj}} & \cdots & \frac{\sum_{j\neq N} r_{Nj}p_{Nj}}{\sum_{j\neq N} r_{Nj}}
\end{pmatrix},
\end{aligned}
$$

where the second equality follows from the definition

$$
r_{ij}\left(\beta,\mathbf{A},W_{ij}\right) = \frac{\exp\left(W'_{ij}\beta\right)}{\exp\left(-A_j\right)+\exp\left(W'_{ij}\beta+A_i\right)} = \exp\left(A_i\right)p_{ij},
$$

and the relationships

$$\frac{\frac{\exp\left(W_{ij}'\beta\right)\exp(-A_j(\beta))}{\left[\exp(-A_j(\beta))+\exp\left(W_{ij}'\beta+A_i(\beta)\right)\right]^2}}{\sum_{j\neq i}\frac{\exp\left(W_{ij}'\beta\right)}{\exp(-A_j(\beta))+\exp\left(W_{ij}'\beta+A_i(\beta)\right)}}=\frac{r_{ij}\left(1-p_{ij}\right)}{\sum_{j\neq i}r_{ij}}=\frac{p_{ij}\left(1-p_{ij}\right)}{\sum_{j\neq i}p_{ij}},$$

and

$$\frac{\sum_{j\neq i}\frac{\exp\left(W_{ij}'\beta\right)\exp\left(W_{ij}'\beta+A_i(\beta)\right)}{\left[\exp(-A_j(\beta))+\exp\left(W_{ij}'\beta+A_i(\beta)\right)\right]^2}}{\sum_{j\neq i}\frac{\exp\left(W_{ij}'\beta\right)}{\exp(-A_j(\beta))+\exp\left(W_{ij}'\beta+A_i(\beta)\right)}}=\frac{\sum_{j\neq i}r_{ij}p_{ij}}{\sum_{j\neq i}r_{ij}}=\frac{\sum_{j\neq i}p_{ij}^2}{\sum_{j\neq i}p_{ij}}.$$

Calculation details for proof of Lemma 6

To derive the bound for R_p appearing in the proof of Lemma 6 observe that

$$\frac{\partial}{\partial\mathbf{A}\partial\mathbf{A}'}s_{\mathbf{A}ij}^{(p)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)=-p_{ij}\left(1-p_{ij}\right)\left(1-2p_{ij}\right)T_{ij}T_{ij}'T_{p,ij}$$

and hence that $\sum_{i=1}^{N}\sum_{j<i}\frac{\partial}{\partial\mathbf{A}\partial\mathbf{A}'}s_{\mathbf{A}ij}^{(p)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)$ equals

$$-\begin{pmatrix}\sum_{j\neq 1}p_{1j}\left(1-p_{1j}\right)\left(1-2p_{1j}\right)T_{p,1j} & \cdots & p_{1N}\left(1-p_{1N}\right)\left(1-2p_{1N}\right)T_{p,1N}\\ \vdots & \ddots & \vdots\\ p_{1N}\left(1-p_{1N}\right)\left(1-2p_{1N}\right)T_{p,1N} & \cdots & \sum_{j\neq N}p_{Nj}\left(1-p_{Nj}\right)\left(1-2p_{Nj}\right)T_{p,Nj}\end{pmatrix}.$$

So that

$$\iota_N'\left[\sum_{i=1}^{N}\sum_{j<i}\frac{\partial}{\partial\mathbf{A}\partial\mathbf{A}'}s_{\mathbf{A}ij}^{(p)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]\iota_N=2\sum_{i=1}^{N}\sum_{j\neq i}p_{ij}\left(1-p_{ij}\right)\left(1-2p_{ij}\right)T_{p,ij}.$$

Finally observe that $\sum_{i=1}^{N}\sum_{j\neq i}T_{p,ij}=2\left(N-1\right).$

Calculation details for proof of Theorem 4

Probability limit of concentrated Hessian: The expression for $H_{N,\beta\beta}+H_{N,\beta\mathbf{A}}V_N^{-1}H_{N,\beta\mathbf{A}}$, the approximate Hessian of the concentrated log-likelihood given in (59), may be calculated as follows

$$
\begin{aligned}
& H_{N,\beta\beta}+H_{N,\beta\mathbf{A}}V_N^{-1}H_{N,\beta\mathbf{A}} \\
= & -\sum_{i=1}^{N}\sum_{j<i}p_{ij}\left(1-p_{ij}\right)W_{ij}W_{ij}' \\
& +\begin{pmatrix} -\sum_{j\neq 1}p_{1j}\left(1-p_{1j}\right)W_{1j} & \cdots & -\sum_{j\neq N}p_{Nj}\left(1-p_{Nj}\right)W_{Nj} \end{pmatrix} \\
& \times\mathrm{diag}\left\{\frac{1}{\sum_{j\neq 1}p_{1j}\left(1-p_{1j}\right)},\ldots,\frac{1}{\sum_{j\neq N}p_{Nj}\left(1-p_{Nj}\right)}\right\}' \\
& \times\begin{pmatrix} -\sum_{j\neq 1}p_{1j}\left(1-p_{1j}\right)W_{1j}' \\ \vdots \\ -\sum_{j\neq N}p_{Nj}\left(1-p_{Nj}\right)W_{Nj}' \end{pmatrix} \\
= & -\sum_{i=1}^{N}\sum_{j<i}p_{ij}\left(1-p_{ij}\right)W_{ij}W_{ij}' \\
& \begin{pmatrix} \frac{-\sum_{j\neq 1}p_{1j}(1-p_{1j})W_{1j}}{\sum_{j\neq 1}p_{1j}(1-p_{1j})} & \cdots & \frac{-\sum_{j\neq N}p_{Nj}\left(1-p_{Nj}\right)W_{Nj}}{\sum_{j\neq N}p_{Nj}\left(1-p_{Nj}\right)} \end{pmatrix}\begin{pmatrix} -\sum_{j\neq 1}p_{1j}\left(1-p_{1j}\right)W_{1j}' \\ \vdots \\ -\sum_{j\neq N}p_{Nj}\left(1-p_{Nj}\right)W_{Nj}' \end{pmatrix} \\
= & -\left\{\sum_{i=1}^{N}\sum_{j<i}p_{ij}\left(1-p_{ij}\right)W_{ij}W_{ij}'-\sum_{i=1}^{N}\frac{\left(\sum_{j\neq i}p_{ij}\left(1-p_{ij}\right)W_{ij}\right)\left(\sum_{j\neq i}p_{ij}\left(1-p_{ij}\right)W_{ij}\right)'}{\sum_{j\neq i}p_{ij}\left(1-p_{ij}\right)}\right\}.
\end{aligned}
$$

Analysis of remainder term in (61): Let $f(v)=\frac{\exp(v)}{1+\exp(v)}$ be the logit function. To bound the third term in (61) I begin by calculating the derivative of $f(v)(1-f(v))(1-2f(v)) = f(v)-3f(v)^2+2f(v)^3$ with respect to v. Using the fact that $f'(v)=f(v)(1-f(v))$ I get

$$
\begin{aligned}
\frac{\partial}{\partial v}\left\{f(v)(1-f(v))(1-2f(v))\right\} &= f(v)(1-f(v))-6f(v)^2(1-f(v))+6f(v)^3(1-f(v)) \\
&= f(v)(1-f(v))\left(1-6f(v)+6f(v)^2\right) \\
&= f(v)(1-f(v))(1-6f(v)(1-f(v))).
\end{aligned}
$$

Using condition (19) then gives

$$
\sup_{1\leq i,j\leq N}\left|p_{ij}\left(1-p_{ij}\right)\left(1-6p_{ij}\left(1-p_{ij}\right)\right)W_{ij}\right|\leq\frac{1}{4}\left(1-6\kappa\left(1-\kappa\right)\right)\times\sup_{w\in\mathbb{W}}|w|.
$$

Expanding the fourth term in (61) I get

$$
\begin{aligned}
&= \sum_{k=1}^{N}\sum_{l=1}^{N}\left(\hat{A}_k(\beta_0)-A_k(\beta_0)\right)\left(\hat{A}_l(\beta_0)-A_l(\beta_0)\right) \\
&\quad \times\left[\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^3}{\partial A_k\partial A_l\partial \mathbf{A}'}s_{\beta ij}\left(\beta_0,\bar{\mathbf{A}}(\beta_0)\right)\right] \\
&= -\sum_{k=1}^{N}\sum_{l\neq k}\begin{pmatrix} 0 \\ \vdots \\ \left(\hat{A}_k-A_k\right)\left(\hat{A}_l-A_l\right)p_{kl}(1-p_{kl})(1-6p_{kl}(1-p_{kl}))W'_{kl} \\ \vdots \\ \left(\hat{A}_k-A_k\right)\left(\hat{A}_l-A_l\right)p_{kl}(1-p_{kl})(1-6p_{kl}(1-p_{kl}))W'_{kl} \\ \vdots \\ 0 \end{pmatrix}' \\
&= -2\begin{pmatrix} \left(\hat{A}_1-A_1\right)\sum_{j\neq 1}\left(\hat{A}_j-A_j\right)p_{1j}(1-p_{1j})(1-6p_{1j}(1-p_{1j}))W'_{1j} \\ \vdots \\ \left(\hat{A}_N-A_N\right)\sum_{j\neq N}\left(\hat{A}_j-A_j\right)p_{Nj}(1-p_{Nj})(1-6p_{Nj}(1-p_{Nj}))W'_{1j} \end{pmatrix}'.
\end{aligned}
$$

Multiplying this by the $N\times 1$ vector $\hat{\mathbf{A}}-\mathbf{A}$ yields the $K\times 1$ vector

-

$$
2\sum_{i=1}^{N}\sum_{j\neq i}\left(\hat{A}_i-A_i\right)^2\left(\hat{A}_j-A_j\right)(1-p_{ij})(1-6p_{ij}(1-p_{ij}))W_{ij}
$$

which gives (62) of the main text.

Derivation of asymptotic bias: To derive (63) it is convenient to proceed regressor by regressor. Observe that the k^{th} element of the third term appearing in (61) is, for $k=1,\ldots,K$,

$$
\frac{1}{2}\frac{1}{\sqrt{n}}\left[\sum_{l=1}^{N}\left(\hat{A}_l(\beta_0)-A_l(\beta_0)\right)\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l\partial \mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}(\beta_0)\right)\right]\left(\hat{\mathbf{A}}(\beta_0)-\mathbf{A}(\beta_0)\right) \quad (68)
$$

The probability limit of (68) equals (63). To simplify (68) and, derive this limit, start by observing that, for $l=1,\ldots,N$,

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l\partial\mathbf{A}'}s_{\beta ij}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right) &= -\sum_{i=1}^{N}\sum_{j<i}p_{ij}\left(1-p_{ij}\right)\left(1-2p_{ij}\right)W_{ij}T_{ij}'T_{l,ij} \\ &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}p_{ij}\left(1-p_{ij}\right)\left(1-2p_{ij}\right)W_{ij}T_{ij}'T_{l,ij} \\ &\quad -\Big(\; p_{1l}\left(1-p_{1l}\right)\left(1-2p_{1l}\right)W_{1l} \\ &\quad\;\; \cdots \quad p_{l-1l}\left(1-p_{l-1l}\right)\left(1-2p_{l-1l}\right)W_{l-1l} \\ &\quad \sum_{j\neq l}p_{lj}\left(1-p_{lj}\right)\left(1-2p_{lj}\right)W_{lj} \\ &\quad p_{l+1l}\left(1-p_{l+1l}\right)\left(1-2p_{l+1l}\right)W_{l+1l} \\ &\quad\;\; \cdots \quad p_{Nl}\left(1-p_{Nl}\right)\left(1-2p_{Nl}\right)W_{Nl}\;\Big).\end{aligned}$$

Next, using (49) from the proof of Lemma 6 and recalling that e_l is a conformable selection vector with a 1 in its l^{th} element and zeros elsewhere, gives

$$\hat{A}_l\left(\beta_0\right)-A_l\left(\beta_0\right)=-e_l'H_{N,\mathbf{AA}}^{-1}\begin{pmatrix}D_{1+}-p_{1+}\\ \vdots\\ D_{N+}-p_{N+}\end{pmatrix}+o_p\left(1\right).$$

which allows the k^{th} element of the third term in (61) to be replaced with its asymptotic equivalent

$$\frac{1}{2}\frac{1}{\sqrt{n}}\left[\sum_{l=1}^{N}\begin{pmatrix}D_{1+}-p_{1+}\\ \vdots\\ D_{N+}-p_{N+}\end{pmatrix}'H_{N,\mathbf{AA}}^{-1}e_l\left\{\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l\partial\mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right\}\right.$$
$$\left.H_{N,\mathbf{AA}}^{-1}\begin{pmatrix}D_{1+}-p_{1+}\\ \vdots\\ D_{N+}-p_{N+}\end{pmatrix}\right]. \tag{69}$$

Applying the trace operator to (69) and cycling elements yields

$$\frac{1}{2}\frac{1}{\sqrt{n}}\sum_{l=1}^{N}\mathrm{Tr}\left(\left\{\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l \partial \mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right\}H_{N,\mathbf{AA}}^{-1}\begin{pmatrix} D_{1+}-p_{1+} \\ \vdots \\ D_{N+}-p_{N+}\end{pmatrix}\begin{pmatrix} D_{1+}-p_{1+} \\ \vdots \\ D_{N+}-p_{N+}\end{pmatrix}' H_{N,\mathbf{AA}}^{-1}e_l\right),$$

which, after taking expectations conditional on $\mathbf{X}$ and $\mathbf{A}_0$, gives

$$-\frac{1}{2}\frac{1}{\sqrt{n}}\sum_{l=1}^{N}\mathrm{Tr}\left(\left\{\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l \partial \mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right\}H_{N,\mathbf{AA}}^{-1}e_l\right) \tag{70}$$

The difference between (69) and its expectation (70) is $o_p(1)$. To see this observe that the diagonal elements of the $N\times N$ matrix

$$\left[\sum_{i=1}^{N}\sum_{j<i}s_{\mathbf{A}ij}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]\left[\sum_{i=1}^{N}\sum_{j<i}s_{\mathbf{A}ij}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]' = \begin{pmatrix} D_{1+}-p_{1+} \\ \vdots \\ D_{N+}-p_{N+}\end{pmatrix}\begin{pmatrix} D_{1+}-p_{1+} \\ \vdots \\ D_{N+}-p_{N+}\end{pmatrix}'. \tag{71}$$

consist of the terms $\left(D_{i+}-p_{i+}\right)^2$ for $i=1,\ldots,N$. Fix i, order the balance of units arbitrarily, and define $l_{j|i}=\left(D_{ij}-p_{ij}\right)\left(D_{i+}-p_{i+}\right)-p_{ij}\left(1-p_{ij}\right)$; note that $\left\{l_{j|i}\right\}_{j=1}^{\infty}$ is a martingale difference sequence (with $\mathbb{E}\left[\left.l_{j|i}\right| l_{1|i},\ldots,l_{j-1|i}\right]=0$ and bounded moments). A law of large numbers for martingale difference sequences therefore gives (recalling that a $+$ denotes summation over the omitted subscript)

$$\frac{1}{N-1}\left(D_{i+}-p_{i+}\right)^2 \overset{p}{\to} \lim_{N\to\infty}\left\{\frac{\sum_{j\neq i}p_{ij}\left(1-p_{ij}\right)}{N-1}\right\}.$$

A similar argument can be used to characterize the probability limits of the off-diagonal elements of (71)

$$\frac{1}{N-1}\left(D_{i+}-p_{i+}\right)\left(D_{k+}-p_{k+}\right) \overset{p}{\to} \lim_{N\to\infty}\left\{\frac{p_{ik}\left(1-p_{ik}\right)}{N-1}\right\}.$$

Together these results imply that $H_{N,\mathbf{AA}}^{-1}\left[\sum_{i=1}^{N}\sum_{j<i}s_{\mathbf{A}ij}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]\left[\sum_{i=1}^{N}\sum_{j<i}s_{\mathbf{A}ij}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]' = -I_N+o_p(1)$ and hence (70).

To evaluate (70) it is convenient to be able to replace $H_{N,\mathbf{AA}}^{-1}$ with $-V_N$:

$$
\begin{aligned}
&-\frac{1}{2\sqrt{n}}\sum_{l=1}^{N}\operatorname{Tr}\left(\left\{\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l\partial\mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right\}H_{N,\mathbf{AA}}^{-1}e_l\right)\\
=\;&\frac{1}{2\sqrt{n}}\sum_{l=1}^{N}\operatorname{Tr}\left(\left[\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l\partial\mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]V_N^{-1}e_l\right)\\
&+\frac{1}{2\sqrt{n}}\sum_{l=1}^{N}\operatorname{Tr}\left(\left[\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l\partial\mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]\left(-H_{N,\mathbf{AA}}^{-1}-Q_N\right)e_l\right)\\
&+\frac{1}{2\sqrt{n}}\sum_{l=1}^{N}\operatorname{Tr}\left(\left[\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l\partial\mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]\left(Q_{N,\mathbf{AA}}-V_N^{-1}\right)e_l\right).
\end{aligned}
\tag{72}
$$

The first term in (72) coincides with the k^{th} element of the bias expression given in the statement of the theorem. Evaluating this term yields

$$
\begin{aligned}
&\frac{1}{2}\frac{1}{\sqrt{n}}\sum_{l=1}^{N}\operatorname{Tr}\left(\left[\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l\partial\mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]V_N^{-1}e_l\right)=\\
&\frac{1}{2}\frac{1}{\sqrt{n}}\sum_{l=1}^{N}\left[\sum_{i=1}^{N}\sum_{j<i}\frac{\partial^2}{\partial A_l\partial\mathbf{A}'}s_{\beta ij}^{(k)}\left(\beta_0,\mathbf{A}\left(\beta_0\right)\right)\right]\\
&\times\begin{pmatrix}0\\ \vdots\\ 0\\ \frac{1}{\sum_{j\neq l}p_{lj}\left(1-p_{lj}\right)}\\ 0\\ \vdots\\ 0\end{pmatrix}=\\
&-\frac{1}{2}\frac{1}{\sqrt{n}}\sum_{l=1}^{N}\frac{\sum_{j\neq l}p_{lj}\left(1-p_{lj}\right)\left(1-2p_{lj}\right)W_{k,lj}}{\sum_{j\neq l}p_{lj}\left(1-p_{lj}\right)}.
\end{aligned}
$$

The second and third terms are asymptotically negligible. Equation (63) follows directly.