Errata in "Econometric Methods for the Analysis of Assignment Problems in the Presence of Complementarity and Social Spillovers"

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March 7, 2014

After publication of my *Handbook of Social Economics* survey chapter Bernard Salanié kindly pointed out a *non sequitur* in my proof of Theorem 4.1 (Graham, 2011). Specifically the assertion in Part 3 of the proof does not follow from the stated assumptions (cf., equations (77) and (78) in the published proof). It turns out, however, that the conclusion of the Theorem can be proved under a weaker implication than that asserted in Part 3 of the published proof. This note details this correction. Notation follows that in the *Handbook* chapter unless stated otherwise.[1]

Parts 1 and 2 of the proof showed strict monotonicity of the firm's and worker's choice probabilities in the deterministic component of match utility following arguments due to Manski (1975) and Fox (2009c). This portion of the proof remains unchanged.

Here I begin with Part 3. Recall that the proof makes use of the notion of a $klmn$ sub-allocation. That is, we look at all type k and m firms that choose to match with a type l or n worker and all type l and n workers who choose to match with a type k or m firms. We are interested in whether the observed assortativeness of this sub-allocation is related to the unobserved structure of the match surplus function. Specifically whether the sign of the increasing difference $\delta_{mn}-\delta_{ml}-(\delta_{kn}-\delta_{kl})$ – a discrete measure of complementarity – predicts the sign of $r^{klmn}-p^{klmn}q^{klmn}$ – a measure of assortativeness – (see Table 1 and Graham, Imbens and Ridder, 2007).

The conditional probabilities that each type of firm chooses each type of worker and vice versa are, using notation from the published proof,

$$
\begin{aligned}
\Pr\left(X_i=x_n\middle| W_i=w_m, X_i\in\{x_l,x_n\}\right) &= F_{\lambda_l-\lambda_n}\left(\delta_{mn}-\delta_{ml}-(\tau_{mn}-\tau_{ml})\middle| W_i=w_m, X_i\in\{x_l,x_n\}\right)\\
\Pr\left(X_i=x_n\middle| W_i=w_k, X_i\in\{x_l,x_n\}\right) &= F_{\lambda_l-\lambda_n}\left(\delta_{kn}-\delta_{kl}-(\tau_{kn}-\tau_{kl})\middle| W_i=w_k, X_i\in\{x_l,x_n\}\right)\\
\Pr\left(W^j=w_m\middle| X^j=x_n, W^j\in\{w_k,w_m\}\right) &= F_{\rho_k-\rho_n}\left(\tau_{mn}-\tau_{kn}\middle| X^j=x_n, W^j\in\{w_k,w_m\}\right)\\
\Pr\left(W^j=w_m\middle| X^j=x_l, W^j\in\{w_k,w_m\}\right) &= F_{\rho_k-\rho_n}\left(\tau_{ml}-\tau_{kl}\middle| X^j=x_l, W^j\in\{w_k,w_m\}\right).
\end{aligned}
$$

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I thank Bernard Salanié for pointing out the error in my proof of Theorem 4.1 in Graham (2011) and for useful discussion. I also thank Jeremy Fox for help comments. All the usual disclaimers apply.

[1] Irritatingly a number of typographical errors were introduced into the manuscript at the typesetting stage and regrettably I did not manage to catch all of them when reviewing the proofs. In the event of confusion regarding notation in the published chapter I suggest consulting the last pre-publication version of the chapter which is available on my web page.

Table 1: The set of feasible $klmn$ sub-allocations

	$X^j = x_l$	$X^j = x_n$	
$W_i = w_k$	r^{klmn}	$p^{klmn} - r^{klmn}$	p^{klmn}
$W_i = w_m$	$q^{klmn} - r^{klmn}$	$1 - p^{klmn} - q^{klmn} + r^{klmn}$	$1 - p^{klmn}$
	q^{klmn}	$1 - q^{klmn}$	

Market clearing, or sub-allocation feasibility (see Table 1), imposes the equalities

$$\begin{aligned} &\left(1 - p^{klmn}\right) F_{\lambda_l - \lambda_n}\left(\delta_{mn} - \delta_{ml} - (\tau_{mn} - \tau_{ml}) \middle| W_i = w_m, X_i \in \{x_l, x_n\}\right) \\ &\quad + p^{klmn} F_{\lambda_l - \lambda_n}\left(\delta_{kn} - \delta_{kl} - (\tau_{kn} - \tau_{kl}) \middle| W_i = w_k, X_i \in \{x_l, x_n\}\right) = 1 - q^{klmn} \end{aligned}$$

and

$$\begin{aligned} &\left(1 - q^{klmn}\right) F_{\rho_k - \rho_n}\left(\tau_{mn} - \tau_{kn} \middle| X^j = x_n, W^j \in \{w_k, w_m\}\right) \\ &\quad + q^{klmn} F_{\rho_k - \rho_n}\left(\tau_{ml} - \tau_{kl} \middle| X^j = x_l, W^j \in \{w_k, w_m\}\right) = 1 - p^{klmn}, \end{aligned}$$

or that (i) total *firm* demand for type n workers and (ii) total *worker* demand for type m firms coincides with the available supplies.

Assume we observe the *random sub-allocation* (i.e., $r^{klmn} = p^{klmn} q^{klmn}$) , then it must be the case that the conditional probability that a type k or m firm choose a type n worker equals $1 - q^{klmn}$, the marginal frequency of type n workers in the sub-allocation. Likewise the conditional probability that a type l or n worker choose a type m firm will be given by $1 - p^{klmn}$, the marginal frequency of type m firms in the sub-allocation. Formally, in a random sub-allocation we have

$$\Pr\left(X_i = x_n \middle| W_i = w_m, X_i \in \{x_l, x_n\}\right) = \Pr\left(X_i = x_n \middle| W_i = w_k, X_i \in \{x_l, x_n\}\right) = 1 - q^{klmn} \quad (1)$$

$$\Pr\left(W^j = w_m \middle| X^j = x_n, W^j \in \{w_k, w_m\}\right) = \Pr\left(W^j = w_m \middle| X^j = x_l, W^j \in \{w_k, w_m\}\right) = 1 - p^{klmn}. \quad (2)$$

Exploiting strict monotonicity of the choice probabilities, shown in Parts 1 and 2 of the published proof, then gives, after inverting,

$$F^{-1}_{\lambda_l - \lambda_n}\left(1 - q^{klmn} \middle| W_i = w_m, X_i \in \{x_l, x_n\}\right) = F^{-1}_{\lambda_l - \lambda_n}\left(1 - q^{klmn} \middle| W_i = w_k, X_i \in \{x_l, x_n\}\right) \quad (3)$$

and hence

$$\delta_{mn} - \delta_{ml} - (\delta_{kn} - \delta_{kl}) = (\tau_{mn} - \tau_{ml}) - (\tau_{kn} - \tau_{kl}). \quad (4)$$

Similarly we have

$$F^{-1}_{\rho_k - \rho_n}\left(1 - p^{klmn} \middle| X^j = x_n, W^j \in \{w_k, w_m\}\right) = F_{\rho_k - \rho_n}\left(1 - p^{klmn} \middle| X^j = x_l, W^j \in \{w_k, w_m\}\right) \quad (5)$$

and hence

$$(\tau_{mn} - \tau_{ml}) - (\tau_{kn} - \tau_{kl}) = 0. \quad (6)$$

Since both (4) and (6) must hold simultaneously at the random sub-allocation, I conclude that the lack of sub-allocation assortativeness (i.e., observing $r^{klmn} = p^{klmn} q^{klmn}$) implies that the local complementarity parameter $\delta_{mn} - \delta_{ml} - (\delta_{kn} - \delta_{kl})$ is identically equal to zero. In the published proof I claimed that (3) and

(5) held globally. The above argument establishes the weaker claim that they hold only under the random sub-allocation (i.e., when $\delta_{mn} - \delta_{ml} - (\delta_{kn} - \delta_{kl}) = 0$). Note that exploiting the feasibility constraints (1) and (2) is an essential component of the argument (in addition to strict monotonicity of the conditional choice probabilities).

The final step of the proof, Part 4, remains essentially unchanged, following the argument developed on pp. 1018 - 1021 of the published chapter. For completeness I provide details here. Exploiting strict monotonicity of the conditional choice probabilities yields the equalities

$$\begin{aligned}
F^{-1}_{\lambda_l-\lambda_n}\left(\frac{1-p^{klmn}-q^{klmn}+r^{klmn}}{1-p^{klmn}}\middle| W_i = w_m, X_i \in \{x_l, x_n\}\right) &= \delta_{mn} - \delta_{ml} - (\tau_{mn} - \tau_{ml}) \\
F^{-1}_{\lambda_l-\lambda_n}\left(\frac{p^{klmn}-r^{klmn}}{p_k}\middle| W_i = w_k, X_i \in \{x_l, x_n\}\right) &= \delta_{kn} - \delta_{kl} - (\tau_{kn} - \tau_{kl}) \\
F^{-1}_{\rho_k-\rho_n}\left(\frac{1-p^{klmn}-q^{klmn}+r^{klmn}}{1-q^{klmn}}\middle| X^j = x_n, W^j \in \{w_k, w_m\}\right) &= \tau_{mn} - \tau_{kn} \\
F^{-1}_{\rho_k-\rho_n}\left(\frac{q^{klmn}-r^{klmn}}{q^{klmn}}\middle| X^j = x_l, W^j \in \{w_k, w_m\}\right) &= \tau_{ml} - \tau_{kl},
\end{aligned}$$

and hence, after summing,

$$\begin{aligned}
&F^{-1}_{\lambda_l-\lambda_n}\left(\frac{1-p^{klmn}-q^{klmn}+r^{klmn}}{1-p^{klmn}}\middle| W_i = w_m, X_i \in \{x_l, x_n\}\right) \\
&-F^{-1}_{\lambda_l-\lambda_n}\left(\frac{p^{klmn}-r^{klmn}}{p_k}\middle| W_i = w_k, X_i \in \{x_l, x_n\}\right) \\
&+F^{-1}_{\rho_k-\rho_n}\left(\frac{1-p^{klmn}-q^{klmn}+r^{klmn}}{1-q^{klmn}}\middle| X^j = x_n, W^j \in \{w_k, w_m\}\right) \\
&-F^{-1}_{\rho_k-\rho_n}\left(\frac{q^{klmn}-r^{klmn}}{q^{klmn}}\middle| X^j = x_l, W^j \in \{w_k, w_m\}\right) = \delta_{mn} - \delta_{ml} - (\delta_{kn} - \delta_{kl}). \quad (7)
\end{aligned}$$

Observe that from Part 3 above the left-hand side of (7) evaluates to zero at $r^{klmn} = p^{klmn}q^{klmn}$. Differentiating (7) with respect to r^{klmn} yields (invoking Assumptions (v) of the published proof)

$$\begin{aligned}
&\frac{1}{f_{\lambda_l-\lambda_n}\left(\frac{1-p^{klmn}-q^{klmn}+r^{klmn}}{1-p^{klmn}}\middle| W_i = w_m, X_i \in \{x_l, x_n\}\right)}\frac{1}{1-p^{klmn}} \\
&+\frac{1}{f_{\lambda_l-\lambda_n}\left(\frac{p^{klmn}-r^{klmn}}{p_k}\middle| W_i = w_k, X_i \in \{x_l, x_n\}\right)}\frac{1}{p^{klmn}} \\
&+\frac{1}{f_{\rho_k-\rho_n}\left(\frac{1-p^{klmn}-q^{klmn}+r^{klmn}}{1-q^{klmn}}\middle| X^j = x_n, W^j \in \{w_k, w_m\}\right)}\frac{1}{1-q^{klmn}} \\
&+\frac{1}{f_{\rho_k-\rho_n}\left(\frac{q^{klmn}-r^{klmn}}{q^{klmn}}\middle| X^j = x_l, W^j \in \{w_k, w_m\}\right)}\frac{1}{q^{klmn}} > 0.
\end{aligned}$$

Since the left-hand-side of (7) is increasing in r^{klmn} and passes through zero at $r^{klmn} = p^{klmn}q^{klmn}$ we have the implication that if $r^{klmn} \lesseqgtr p^{klmn}q^{klmn}$ we may conclude that $\delta_{mn} - \delta_{ml} - (\delta_{kn} - \delta_{kl}) \lesseqgtr 0$ as claimed.

As in other "maximum score" type identification results (e.g., Manski, 1975; 1987), strict monotonicity of the conditional choice probabilities plays an essential role in the argument. The novelty here is in exploiting the implications of market clearing to justify the assertion that a lack of local assortativeness in the matching implies an absence of complementarity in the surplus function. This intuition was developed heuristically in Fox (2010) (i.e., without a primitive data generating process to justify it). Theorem 4.1 shows that the inference that (local) assortativeness in the allocation implies (local) complementarity in the surplus function holds in a semiparametric generalization of the Choo and Siow (2006a,b) model. To develop this point further observe that under the Type 1 extreme value assumption the left-hand-side of (7) evaluates to

$$\ln\left(\frac{1-p^{klmn}-q^{klmn}+r^{klmn}}{q^{klmn}-r^{klmn}}\frac{r^{klmn}}{p^{klmn}-r^{klmn}}\right),$$

which is a log odds ratio, a well-known measure of positive association from contingency table analysis (cf., Siow, 2009; Galichon and Salanié, 2009).

Let i and j index two independent random draws from the distribution of matches. Recall the definitions: $S_{ij} = \mathrm{sgn}\left\{(W_i - W_j)(X_i - X_j)\right\}$, $\mathbf{A}_{ij}$ the vector of sub-allocation indicators, and ϕ the conformable vector of local complementarity parameters. From the argument given on the top of p. 1021 (cf., Manski, 1987) we have

$$\mathrm{med}\left(S_{ij}\,|\,S_{ij}\in\{-1,1\},\mathbf{A}_{ij}\right)=\mathrm{sgn}\left\{\mathbf{A}_{ij}'\phi\right\},$$

which leads to a maximum score type criterion function.

References

[1] Fox, Jeremy. (2010). "Identification in matching games," *Quantitative Economics* 1 (2): 203 -254.

[2] Graham, Bryan S. (2011). "Econometric methods for the analysis of assignment problems in the presence of complementarity and social spillovers," *Handbook of Social Economics* 1B: 965 - 1052 (J. Benhabib, A. Bisin & M.O. Jackons, Eds.). Amsterdam: North-Holland.