Detailed derivations for omitted proof steps

This appendix includes tedious algebraic details of the various proofs outlined in the Appendix to the main paper and the Supplemental Web Appendix. In what follows 'main appendix' refers to both of these appendices. Equation numbering continues in sequence with that established in the main text, its appendix, and the supplement.

Details of derivation of limiting variance of IPW estimator (Proposition 2.1) To derive (55) we first compute the inverse of (54)

$$M^{-1} = \begin{pmatrix} \Gamma^{-1} & -\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathcal{I}\left(\delta_0\right)^{-1} \\ 0 & -\mathcal{I}\left(\delta_0\right)^{-1} \end{pmatrix},$$

and using (53) multiply out

$$\begin{aligned} M^{-1}\Omega M^{-1\prime} &= \begin{pmatrix} \Gamma^{-1} & -\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathcal{I}\left(\delta_0\right)^{-1} \\ 0 & -\mathcal{I}\left(\delta_0\right)^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{E}\left[\frac{\psi\psi'}{G}\right] & \mathbb{E}\left[\frac{G_1}{G}\psi t'\right] \\ \mathbb{E}\left[\frac{G_1}{G}t\psi'\right] & \mathcal{I}\left(\delta\right) \end{pmatrix} \begin{pmatrix} \Gamma^{-1} & -\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathcal{I}\left(\delta_0\right)^{-1} \\ 0 & -\mathcal{I}\left(\delta_0\right)^{-1} \end{pmatrix}' \\ &= \begin{pmatrix} \Gamma^{-1}\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - \Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right] & 0 \\ -\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right] & -I_{1+M} \end{pmatrix} \begin{pmatrix} \Gamma^{-1\prime} & 0 \\ -\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Gamma^{-1\prime} & -\mathcal{I}\left(\delta_0\right)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \Gamma^{-1}\mathbb{E}\left[\frac{\psi\psi'}{G}\right]\Gamma^{-1\prime} - \Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Gamma^{-1\prime} & 0 \\ 0 & \mathcal{I}\left(\delta_0\right)^{-1} \end{pmatrix}. \end{aligned}$$

Equation (56) may be derived by noting that

$$\mathbb{E}\left[\frac{G_1}{G}\psi t'\right] = \mathbb{E}\left[\frac{D}{G}\psi S_\delta'\right],$$

for $S_\delta = \frac{D-G}{G(1-G)}G_1 t$. Let $\Pi_S = \mathbb{E}\left[\frac{D}{G}\psi S_\delta'\right]\mathcal{I}\left(\delta_0\right)^{-1}$ as in the main text; manipulation gives

$$\begin{aligned} &\Gamma^{-1}\mathbb{E}\left[\frac{\psi\psi'}{G}\right]\Gamma^{-1\prime} - \Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Gamma^{-1\prime} \\ &= \Gamma^{-1}\mathbb{E}\left[\frac{D}{G}\psi\frac{D}{G}\psi'\right]\Gamma^{-1\prime} - \Gamma^{-1}\mathbb{E}\left[\frac{D}{G}\psi S_\delta'\right]\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[S_\delta\frac{D}{G}\psi'\right]\Gamma^{-1\prime} \\ &= \Gamma^{-1}\left\{\mathbb{E}\left[\left(\frac{D}{G}\psi - \Pi_S S_\delta\right)\left(\frac{D}{G}\psi - \Pi_S S_\delta\right)'\right]\right\}\Gamma^{-1\prime} \\ &= \Gamma^{-1}\left\{\mathbb{E}\left[\left(\frac{D}{G}\psi - \left(\frac{D}{G}-1\right)q + \left\{\left(\frac{D}{G}-1\right)q - \Pi_S S_\delta\right\}\right)\left(\frac{D}{G}\psi - \left(\frac{D}{G}-1\right)q + \left\{\left(\frac{D}{G}-1\right)q - \Pi_S S_\delta\right\}\right)'\right]\right\}\Gamma^{-1\prime} \\ &= \mathcal{I}\left(\gamma_0\right)^{-1} + \Gamma^{-1}\mathbb{E}\left[\left(\left(\frac{D}{G}-1\right)q - \Pi_S S_\delta\right)\left(\left(\frac{D}{G}-1\right)q - \Pi_S S_\delta\right)'\right]\Gamma^{-1\prime}, \end{aligned}$$

where the last line follows from the fact that

$$
\begin{aligned}
&\mathbb{E}\left[\left(\frac{D}{G}\psi-\left(\frac{D}{G}-1\right)q\right)\left(\frac{D}{G}\psi-\left(\frac{D}{G}-1\right)q\right)'\right]\\
&=\mathbb{E}\left[\frac{D}{G^2}\psi\psi-\left(\frac{D}{G}-1\right)\frac{D}{G}\psi q'-\left(\frac{D}{G}-1\right)\frac{D}{G}q\psi'+\left(\frac{D}{G}-1\right)^2qq'\right]\\
&=\mathbb{E}\left[\frac{\mathbb{E}\left[\psi\psi|\,X\right]}{G}-\frac{1-G}{G}qq'\right]\\
&=\Lambda_0
\end{aligned}
$$

and also

$$
\begin{aligned}
&\mathbb{E}\left[\left\{\frac{D}{G}\psi-\left(\frac{D}{G}-1\right)q\right\}\left\{\left(\frac{D}{G}-1\right)q-\Pi_S S_\delta\right\}'\right]\\
&=\mathbb{E}\left[\left(\frac{D}{G}-1\right)\frac{D}{G}\psi q'\right]-\mathbb{E}\left[\left(\frac{D}{G}-1\right)^2qq'\right]-\mathbb{E}\left[\frac{D}{G}\psi S_\delta'\Pi_S'\right]+\mathbb{E}\left[\left(\frac{D}{G}-1\right)qS_\delta'\Pi_S'\right]\\
&=\mathbb{E}\left[\frac{1-G}{G}qq'\right]-\mathbb{E}\left[\frac{1-G}{G}qq'\right]\\
&-\mathbb{E}\left[\frac{D}{G}\psi S_\delta'\right]\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[S_\delta\frac{D}{G}\psi'\right]+\mathbb{E}\left[\frac{D}{G}\psi S_\delta'\right]\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[S_\delta\frac{D}{G}\psi'\right]-0\\
&=0,
\end{aligned}
$$

with the second equality making using of the fact that $\mathbb{E}\left[S_\delta|\,X\right]=0$.

Details of derivation of limiting variance of IPT estimator (Theorems 2.2 and 2.1) To derive (30) we first compute the inverse of M as

$$
M^{-1}=\begin{pmatrix}\Gamma^{-1} & -\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\\ 0 & -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\end{pmatrix},
$$

and then multiply out:

$$M^{-1}\Omega M^{-1\prime} = \begin{pmatrix} \Gamma^{-1} & -\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \\ 0 & -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{E}\left[\frac{\psi\psi'}{G}\right] & E_0 \\ E_0' & F_0 \end{pmatrix}$$

$$\times \begin{pmatrix} \Gamma^{-1} & -\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \\ 0 & -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \end{pmatrix}'$$

$$= \begin{pmatrix} \Gamma^{-1}\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - \Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}E_0' & \Gamma^{-1}E_0 - \Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}F_0 \\ -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}E_0' & -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}F_0 \end{pmatrix}$$

$$\times \begin{pmatrix} \Gamma^{-1\prime} & 0 \\ -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Gamma^{-1\prime} & -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} \Gamma^{-1}\mathbb{E}\left[\frac{\psi\psi'}{G}\right]\Gamma^{-1\prime} - \Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}E_0'\Gamma^{-1\prime} \\ -\Gamma^{-1}E_0\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Gamma^{-1\prime} + \Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}F_0\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Gamma^{-1\prime} \end{pmatrix} \\ -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}E_0'\Gamma^{-1\prime} + \mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}F_0\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Gamma^{-1\prime} \end{pmatrix}$$

$$\begin{matrix} -\Gamma^{-1}E_0\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} + \Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}F_0\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \\ \mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}F_0\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \end{matrix} \Bigg).$$

Manipulating the upper-left-hand block of this matrix we get

$$\begin{aligned}
&\Gamma^{-1}\mathbb{E}\left[\frac{\psi\psi'}{G}\right]\Gamma^{-1\prime} \\
&-\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}E_0'\Gamma^{-1\prime} \\
&-\Gamma^{-1}E_0\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Gamma^{-1\prime} \\
&+\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}F_0\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Gamma^{-1\prime} \\
&=\Gamma^{-1}\mathbb{E}\left[\frac{\psi\psi'}{G}\right]\Gamma^{-1\prime} - \Gamma^{-1}E_0F_0^{-1}E_0'\Gamma^{-1\prime} \\
&+\Gamma^{-1}\Bigg\{E_0F_0^{-1}E_0' - \mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}E_0' \\
&-E_0\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right] + \mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}F_0\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Bigg\}\Gamma^{-1\prime} \\
&=\Gamma^{-1}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_0F_0^{-1}E_0'\right)\Gamma^{-1\prime} \\
&+\Gamma^{-1}\left(E_0F_0^{-1} - \mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\right)F_0\left(E_0F_0^{-1} - \mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\right)'\Gamma^{-1},
\end{aligned}$$

Recall that $\Delta_0 = \mathbb{E}\left[\frac{G_1}{G}\psi t'\right] - E_0F_0^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]$ so that

$$\Delta_0\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} = \mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} - E_0F_0^{-1}$$

which gives an upper-left-hand block equal to

$$\begin{aligned}
&\Gamma^{-1}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_0F_0^{-1}E_0'\right)\Gamma^{-1\prime}\\
&+\Gamma^{-1}\left(E_0F_0^{-1}-\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\right)F_0\left(E_0F_0^{-1}-\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\right)'\Gamma^{-1}\\
&=\Gamma^{-1}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_0F_0^{-1}E_0'\right)\Gamma^{-1\prime}\\
&+\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\Delta_0'F_0\Delta_0\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\Gamma^{-1\prime}.
\end{aligned}$$

We can rearrange the off-diagonal blocks as follows

$$\begin{aligned}
&-\Gamma^{-1}E_0\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}+\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}F_0\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\\
&=-\Gamma^{-1}\left\{E_0F_0^{-1}-\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\right\}F_0\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}.
\end{aligned}$$

Proof of Lemma A.1 Define the partition

$$\begin{aligned}
\overline{V}(\beta)&=\begin{pmatrix}
\frac{1}{N}\sum_{i=1}^{N}\frac{D_i}{G_i(\delta)^2}\psi_i(\gamma)\psi_i(\gamma)' & \frac{1}{N}\sum_{i=1}^{N}\frac{D_i}{G_i(\delta)}\omega_i(\delta)\psi_i(\gamma)t_i' & 0\\
\frac{1}{N}\sum_{i=1}^{N}\frac{D_i}{G_i(\delta)}\omega_i(\delta)t_i\psi_i(\gamma)' & \frac{1}{N}\sum_{i=1}^{N}\nu_i(\delta)\omega_i(\delta)t_it_i' & 0\\
0 & \frac{1}{N}\sum_{i=1}^{N}\frac{D_i}{G_i(\delta)}\frac{G_{1i}(\delta)}{G_i(\delta)}t_it_i' & -\frac{1}{N}\sum_{i=1}^{N}J_i(\delta)
\end{pmatrix}\\
&=\begin{pmatrix}
\overline{V}_{11}(\beta) & \overline{V}_{12}(\beta) & 0\\
\overline{V}_{12}(\beta)' & \overline{V}_{22}(\beta) & 0\\
0 & \overline{V}_{32}(\beta) & \overline{V}_{33}(\beta)
\end{pmatrix},
\end{aligned}$$

with $\overline{M}(\beta)$ partitioned similarly. Note that $\overline{V}_{32}(\beta)=-\overline{M}_{22}(\beta)$ and $\overline{V}_{33}(\beta)=-\overline{M}_{32}(\beta)$, equalities that will be exploited below. Using the partitioned inverse formula we get

$$\begin{aligned}
&\overline{V}(\beta)^{-1}\\
&=\left(\begin{matrix}
\left(\overline{V}_{11}(\beta)-\overline{V}_{12}(\beta)\overline{V}_{22}(\beta)^{-1}\overline{V}_{12}(\beta)'\right)^{-1}\\
-\overline{V}_{22}(\beta)^{-1}\overline{V}_{12}(\beta)'\left(\overline{V}_{11}(\beta)-\overline{V}_{12}(\beta)\overline{V}_{22}(\beta)^{-1}\overline{V}_{12}(\beta)'\right)^{-1}\\
\overline{V}_{33}(\beta)^{-1}\overline{V}_{32}(\beta)\overline{V}_{22}(\beta)^{-1}\overline{V}_{12}(\beta)'\left(\overline{V}_{11}(\beta)-\overline{V}_{12}(\beta)\overline{V}_{22}(\beta)^{-1}\overline{V}_{12}(\beta)'\right)^{-1}
\end{matrix}\right.\\
&\qquad\left.\begin{matrix}
-\left(\overline{V}_{11}(\beta)-\overline{V}_{12}(\beta)\overline{V}_{22}(\beta)^{-1}\overline{V}_{12}(\beta)'\right)^{-1}\overline{V}_{12}(\beta)\overline{V}_{22}(\beta)^{-1} & 0\\
\left(\overline{V}_{22}(\beta)-\overline{V}_{12}(\beta)'\overline{V}_{11}(\beta)^{-1}\overline{V}_{12}(\beta)\right)^{-1} & 0\\
-\overline{V}_{33}(\beta)^{-1}\overline{V}_{32}(\beta)\left(\overline{V}_{22}(\beta)-\overline{V}_{12}(\beta)'\overline{V}_{11}(\beta)^{-1}\overline{V}_{12}(\beta)\right)^{-1} & \overline{V}_{33}(\beta)^{-1}
\end{matrix}\right).
\end{aligned}$$

Using this expression we can evaluate $\overline{M}(\beta)' \overline{V}(\beta)^{-1} \overline{m}(\beta)$ as follows

$$
\begin{aligned}
&\overline{M}(\beta)' \overline{V}(\beta)^{-1} \overline{m}(\beta) \\
&= \left(\begin{array}{c}
\overline{M}_{11}(\beta)' \left(\overline{V}_{11}(\beta) - \overline{V}_{12}(\beta) \overline{V}_{22}(\beta)^{-1} \overline{V}_{12}(\beta)'\right)^{-1} \\
\left(\begin{array}{c}
\overline{M}_{12}(\beta)' \left(\overline{V}_{11}(\beta) - \overline{V}_{12}(\beta) \overline{V}_{22}(\beta)^{-1} \overline{V}_{12}(\beta)'\right)^{-1} \\
-\overline{M}_{22}(\beta)' \overline{V}_{22}(\beta)^{-1} \overline{V}_{12}(\beta)' \left(\overline{V}_{11}(\beta) - \overline{V}_{12}(\beta) \overline{V}_{22}(\beta)^{-1} \overline{V}_{12}(\beta)'\right)^{-1} \\
+\overline{M}_{32}(\beta)' \overline{V}_{33}(\beta)^{-1} \overline{V}_{32}(\beta) \overline{V}_{22}(\beta)^{-1} \overline{V}_{12}(\beta)' \left(\overline{V}_{11}(\beta) - \overline{V}_{12}(\beta) \overline{V}_{22}(\beta)^{-1} \overline{V}_{12}(\beta)'\right)^{-1}
\end{array} \right)
\end{array} \right. \\
&\qquad \left. \begin{array}{cc}
-\overline{M}_{11}(\beta)' \left(\overline{V}_{11}(\beta) - \overline{V}_{12}(\beta) \overline{V}_{22}(\beta)^{-1} \overline{V}_{12}(\beta)'\right)^{-1} \overline{V}_{12}(\beta) \overline{V}_{22}(\beta)^{-1} & 0 \\
\left(\begin{array}{c}
-\overline{M}_{12}(\beta)' \left(\overline{V}_{11}(\beta) - \overline{V}_{12}(\beta) \overline{V}_{22}(\beta)^{-1} \overline{V}_{12}(\beta)'\right)^{-1} \overline{V}_{12}(\beta) \overline{V}_{22}(\beta)^{-1} \\
+\overline{M}_{22}(\beta)' \left(\overline{V}_{22}(\beta) - \overline{V}_{12}(\beta)' \overline{V}_{11}(\beta)^{-1} \overline{V}_{12}(\beta)\right)^{-1} \\
-\overline{M}_{32}(\beta)' \overline{V}_{33}(\beta)^{-1} \overline{V}_{32}(\beta) \left(\overline{V}_{22}(\beta) - \overline{V}_{12}(\beta)' \overline{V}_{11}(\beta)^{-1} \overline{V}_{12}(\beta)\right)^{-1}
\end{array} \right) & \overline{M}_{32}(\beta)' \overline{V}_{33}(\beta)^{-1}
\end{array} \right) \times \left(\begin{array}{c} \overline{m}_1(\beta) \\ \overline{m}_2(\beta) \\ \overline{m}_3(\beta) \end{array} \right) \\
&= \left(\begin{array}{c}
\overline{M}_{11}(\beta)' \left(\overline{V}_{11}(\beta) - \overline{V}_{12}(\beta) \overline{V}_{22}(\beta)^{-1} \overline{V}_{12}(\beta)'\right)^{-1} \left[\overline{m}_1(\beta) - \overline{V}_{12}(\beta) \overline{V}_{22}(\beta)^{-1} \overline{m}_2(\beta)\right] \\
\left(\begin{array}{c}
\overline{M}_{12}(\beta)' \times \left(\overline{V}_{11}(\beta) - \overline{V}_{12}(\beta) \overline{V}_{22}(\beta)^{-1} \overline{V}_{12}(\beta)'\right)^{-1} \left[\overline{m}_1(\beta) - \overline{V}_{12}(\beta) \overline{V}_{22}(\beta)^{-1} \overline{m}_2(\beta)\right] \\
-\overline{m}_3(\beta)
\end{array} \right)
\end{array} \right),
\end{aligned}
$$

where we make use of the equalities

$$
\overline{M}_{22}(\beta)' - \overline{M}_{32}(\beta)' \overline{V}_{33}(\beta)^{-1} \overline{V}_{32}(\beta) = 0, \overline{M}_{32}(\beta)' \overline{V}_{33}(\beta)^{-1} = I,
$$

which follow from $\overline{V}_{32}(\beta) = -\overline{M}_{22}(\beta)$ and $\overline{V}_{33}(\beta) = -\overline{M}_{32}(\beta)$.

Now consider the solution to $\overline{M}\left(\widehat{\beta}\right)' \overline{V}\left(\widehat{\beta}\right)^{-1} \overline{m}\left(\widehat{\beta}\right) = 0$. The first block of this vector

$$
\overline{M}_{11}\left(\widehat{\beta}\right)' \left(\overline{V}_{11}\left(\widehat{\beta}\right) - \overline{V}_{12}\left(\widehat{\beta}\right) \overline{V}_{22}\left(\widehat{\beta}\right)^{-1} \overline{V}_{12}\left(\widehat{\beta}\right)'\right)^{-1} \left[\overline{m}_1\left(\widehat{\beta}\right) - \overline{V}_{12}\left(\widehat{\beta}\right) \overline{V}_{22}\left(\widehat{\beta}\right)^{-1} \overline{m}_2\left(\widehat{\beta}\right)\right] = 0,
$$

yields, for $\delta = \widehat{\delta}$ (i.e., the CMLE of δ_0), exactly K equations for the $K \times 1$ vector $\widehat{\gamma}$. The solution to these equations is identical to that of $\overline{m}_1\left(\widehat{\beta}\right) - \overline{V}_{12}\left(\widehat{\beta}\right) \overline{V}_{22}\left(\widehat{\beta}\right)^{-1} \overline{m}_2\left(\widehat{\beta}\right) = 0$. Since this equality must hold at $\beta = \widehat{\beta}$, the second block of $\overline{M}\left(\widehat{\beta}\right)' \overline{V}\left(\widehat{\beta}\right)^{-1} \overline{m}\left(\widehat{\beta}\right) = 0$ equals $-\overline{m}_3\left(\widehat{\beta}\right) = 0$ which implies that the second element of $\widehat{\beta}$ is indeed the MLE of δ_0.

Details of derivation of asymptotic variance of three-step AIPW estimator To derive (64) we begin by using the partitioned inverse formula and tedious manipulation to compute V^{-1} :

$$V^{-1}=\begin{bmatrix} \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1} & -\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1}E_\omega F_\omega^{-1} & 0 \\ -F_\omega^{-1}E_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1} & \left(F_\omega-E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} & 0 \\ \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]F_\omega^{-1}E_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1} & -\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega-E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} & \mathcal{I}(\delta_0)^{-1}\end{bmatrix}. \tag{93}$$

Equation (93) is derived as follows. First the partitioned inverse formula gives

$$\begin{aligned}&\begin{pmatrix}\mathbb{E}\left[\frac{\psi\psi'}{G}\right] & E_\omega \\ E_\omega' & F_\omega\end{pmatrix}^{-1}\\ &=\begin{pmatrix}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1} & -\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\left(F_\omega-E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} \\ -\left(F_\omega-E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1} & \left(F_\omega-E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}\end{pmatrix},\end{aligned}$$

so that

$$V^{-1}=\begin{bmatrix} \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1} & -\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\left(F_\omega-E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} & 0 \\ -\left(F_\omega-E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1} & \left(F_\omega-E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} & 0 \\ \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega-E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1} & -\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega-E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} & \mathcal{I}(\delta_0)^{-1}\end{bmatrix}.$$

This follows since

$$\begin{aligned}&-\mathcal{I}(\delta_0)^{-1}\begin{pmatrix}0 & \mathbb{E}\left[\frac{G_1}{G}tt'\right]\end{pmatrix}\\ &\times\begin{pmatrix}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1} & -\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\left(F_\omega-E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} \\ -\left(F_\omega-E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1} & \left(F_\omega-E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}\end{pmatrix}\\ &=\begin{pmatrix}\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega-E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1} & -\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega-E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}\end{pmatrix}.\end{aligned}$$

Equation (93) then follows after exploiting the additional equality (cf., Henderson and Searle, 1981, Eq. 17).

$$\begin{aligned}
&E_\omega \left(F_\omega - E_\omega' \mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1} E_\omega \right)^{-1} \\
&= E_\omega F_\omega^{-1} + E_\omega F_\omega^{-1} E_\omega' \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega' \right)^{-1} E_\omega F_\omega^{-1} \\
&= \left[\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega' \right) + E_\omega F_\omega^{-1} E_\omega' \right] \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega' \right)^{-1} E_\omega F_\omega^{-1} \\
&= \mathbb{E}\left[\frac{\psi\psi'}{G}\right] \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega' \right)^{-1} E_\omega F_\omega^{-1}.
\end{aligned}$$

We then calculate $M'V^{-1}$ as follows

$$\begin{aligned}
&M'V^{-1} \qquad (94) \\
&= \begin{bmatrix} \Gamma & -\mathbb{E}\left[\frac{G_1}{G}\psi t'\right] \\ 0 & -\mathbb{E}\left[\frac{G_1}{G}tt'\right] \\ 0 & -\mathcal{I}(\delta_0) \end{bmatrix}' V^{-1} \\
&= \begin{pmatrix} \Gamma' & 0 & 0 \\ -\mathbb{E}\left[\frac{G_1}{G}t\psi'\right] & -\mathbb{E}\left[\frac{G_1}{G}tt'\right] & -\mathcal{I}(\delta_0) \end{pmatrix} \\
&\times \begin{bmatrix} \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega' \right)^{-1} & -\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega' \right)^{-1} E_\omega F_\omega^{-1} & 0 \\ -F_\omega^{-1} E_\omega' \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega' \right)^{-1} & \left(F_\omega - E_\omega' \mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1} E_\omega \right)^{-1} & 0 \\ \mathcal{I}(\delta_0)^{-1} \mathbb{E}\left[\frac{G_1}{G}tt'\right] F_\omega^{-1} E_\omega' \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega' \right)^{-1} & -\mathcal{I}(\delta_0)^{-1} \mathbb{E}\left[\frac{G_1}{G}tt'\right] \left(F_\omega - E_\omega' \mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1} E_\omega \right)^{-1} & \mathcal{I}(\delta_0)^{-1} \end{bmatrix} \\
&= \begin{pmatrix} \Gamma' \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega' \right)^{-1} & -\Gamma' \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega' \right)^{-1} E_\omega F_\omega^{-1} & 0 \\ -\mathbb{E}\left[\frac{G_1}{G}t\psi'\right] \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega' \right)^{-1} & \mathbb{E}\left[\frac{G_1}{G}t\psi'\right] \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega' \right)^{-1} E_\omega F_\omega^{-1} & -I_{1+M} \end{pmatrix}.
\end{aligned}$$

Post-multiplying by M gives $M'V^{-1}M$

$$
\begin{aligned}
&= M'V^{-1}M \\
&= \begin{pmatrix} \Gamma'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1} & -\Gamma'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1} E_\omega F_\omega^{-1} & 0 \\ -\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1} & \mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1} E_\omega F_\omega^{-1} & -I_{1+M} \end{pmatrix} \\
&\times \begin{bmatrix} \Gamma & -\mathbb{E}\left[\frac{G_1}{G}\psi t'\right] \\ 0 & -\mathbb{E}\left[\frac{G_1}{G}tt'\right] \\ 0 & -\mathcal{I}(\delta_0) \end{bmatrix} \\
&= \left(\begin{matrix} \Gamma'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Gamma \\ -\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Gamma \end{matrix} \right. \\
&\qquad \left. \begin{matrix} -\Gamma'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right] + \Gamma'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}E_\omega F_\omega^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right] \\ \mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right] - \mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}E_\omega F_\omega^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right] + \mathcal{I}(\delta_0) \end{matrix} \right) \\
&= \begin{pmatrix} \Gamma'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Gamma & -\Gamma'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Delta_\omega \\ -\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Gamma & \mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Delta_\omega + \mathcal{I}(\delta_0) \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_\omega &= \mathbb{E}\left[\frac{G_1}{G}\psi t'\right] - E_\omega F_\omega^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right] \\
&= \mathbb{E}\left[\left\{\psi - E_\omega F_\omega^{-1}t\right\}\frac{G_1}{G}t'\right] \\
&= \mathbb{E}\left[\frac{D}{G}\left\{\psi - E_\omega F_\omega^{-1}t\right\}S_\delta'\right].
\end{aligned}
$$

The partitioned inverse formula then gives

$$
\left(M'V^{-1}M\right)^{-1} = \begin{pmatrix} \Gamma^{-1}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)\Gamma^{-1\prime} + \Gamma^{-1}\Delta_\omega\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Gamma^{-1\prime} & \Gamma^{-1}\Delta_\omega\mathcal{I}(\delta_0)^{-1} \\ \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Gamma^{-1\prime} & \mathcal{I}(\delta_0)^{-1} \end{pmatrix}.
$$

Note that this corresponds to equation (70) in the main appendix after noting that $\Pi_S' = \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]$ and $\Delta_\omega = 0$ under Assumption 2.1.

We now evaluate $\left(M'V^{-1}M\right)^{-1}M'V^{-1}$ as follows:

$$
\begin{aligned}
&\left(M'V^{-1}M\right)^{-1}M'V^{-1} \\
&=\begin{pmatrix} \Gamma^{-1}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)\Gamma^{-1\prime}+\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Gamma^{-1\prime} & \Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1} \\ \mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Gamma^{-1\prime} & \mathcal{I}\left(\delta_0\right)^{-1} \end{pmatrix} \\
&\times\begin{pmatrix} \Gamma'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1} & -\Gamma'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1}E_\omega F_\omega^{-1} & 0 \\ -\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1} & \mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1}E_\omega F_\omega^{-1} & -I_{1+M} \end{pmatrix} \\
&=\left(\begin{matrix} \begin{pmatrix} \left\{\Gamma^{-1}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)\Gamma^{-1\prime}+\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Gamma^{-1\prime}\right\}\Gamma'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1} \\ -\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1} \end{pmatrix} \\ 0 \end{matrix}\right. \\
&\quad\begin{matrix} \begin{pmatrix} -\left\{\Gamma^{-1}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)\Gamma^{-1\prime}+\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Gamma^{-1\prime}\right\}\Gamma'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1}E_\omega F_\omega^{-1} \\ +\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1}E_\omega F_\omega^{-1} \end{pmatrix} \\ 0 \end{matrix} \\
&\quad\left.\begin{matrix} -\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1} \\ -\mathcal{I}\left(\delta_0\right)^{-1} \end{matrix}\right) \\
&=\begin{pmatrix} \Gamma^{-1} & -\Gamma^{-1}E_\omega F_\omega^{-1} & -\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1} \\ 0 & 0 & -\mathcal{I}\left(\delta_0\right)^{-1} \end{pmatrix},
\end{aligned}
\tag{95}
$$

which gives (71) of the main appendix after noting that $E_\omega F_\omega^{-1}=\Pi_0$ and $\Delta_\omega=0$ under Assumption 2.1.

Using this result we compute the limiting variance of the three-step AIPW estimator as

$$
\begin{aligned}
&\left(M'V^{-1}M\right)^{-1} M'V^{-1}\Omega V^{-1}M\left(M'V^{-1}M\right)^{-1\prime} \\
&= \begin{pmatrix} \Gamma^{-1} & -\Gamma^{-1}E_{\omega}F_{\omega}^{-1} & -\Gamma^{-1}\Delta_{\omega}\mathcal{I}\left(\delta_0\right)^{-1} \\ 0 & 0 & -\mathcal{I}\left(\delta_0\right)^{-1} \end{pmatrix} \\
&\times \begin{pmatrix} \mathbb{E}\left[\frac{\psi\psi'}{G}\right] & E_0 & \mathbb{E}\left[\frac{G_1}{G}\psi t'\right] \\ E_0' & F_0 & \mathbb{E}\left[\frac{G_1}{G}tt'\right] \\ \mathbb{E}\left[\frac{G_1}{G}t\psi'\right] & \mathbb{E}\left[\frac{G_1}{G}tt'\right] & \mathcal{I}\left(\delta_0\right) \end{pmatrix} \\
&\times \begin{pmatrix} \Gamma^{-1\prime} & 0 \\ -F_{\omega}^{-1}E_{\omega}'\Gamma^{-1\prime} & 0 \\ -\mathcal{I}\left(\delta_0\right)^{-1}\Delta_{\omega}'\Gamma^{-1\prime} & -\mathcal{I}\left(\delta_0\right)^{-1} \end{pmatrix} \\
&= \left(\begin{matrix} \Gamma^{-1}\left\{\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_{\omega}F_{\omega}^{-1}E_0' - \Delta_{\omega}\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\right\} & \Gamma^{-1}\left\{E_0 - E_{\omega}F_{\omega}^{-1}F_0 - \Delta_{\omega}\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\right\} \\ -\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right] & -\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right] \end{matrix}\right. \\
&\qquad \left. \begin{matrix} \Gamma^{-1}\left\{\mathbb{E}\left[\frac{G_1}{G}\psi t'\right] - E_{\omega}F_{\omega}^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right] - \Delta_{\omega}\right\} \\ -I_{1+M} \end{matrix} \right) \\
&\times \begin{pmatrix} \Gamma^{-1\prime} & 0 \\ -F_{\omega}^{-1}E_{\omega}'\Gamma^{-1\prime} & 0 \\ -\mathcal{I}\left(\delta_0\right)^{-1}\Delta_{\omega}'\Gamma^{-1\prime} & -\mathcal{I}\left(\delta_0\right)^{-1} \end{pmatrix} \\
&= \left(\begin{matrix} \begin{pmatrix} \Gamma^{-1}\left\{\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_{\omega}F_{\omega}^{-1}E_0' - \Delta_{\omega}\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\right\}\Gamma^{-1\prime} \\ -\Gamma^{-1}\left\{E_0 - E_{\omega}F_{\omega}^{-1}F_0 - \Delta_{\omega}\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\right\}F_{\omega}^{-1}E_{\omega}'\Gamma^{-1\prime} \\ -\Gamma^{-1}\left\{\mathbb{E}\left[\frac{G_1}{G}\psi t'\right] - E_{\omega}F_{\omega}^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right] - \Delta_{\omega}\right\}\mathcal{I}\left(\delta_0\right)^{-1}\Delta_{\omega}'\Gamma^{-1\prime} \end{pmatrix} \\ -\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Gamma^{-1\prime} + \mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]F_{\omega}^{-1}E_{\omega}'\Gamma^{-1\prime} + \mathcal{I}\left(\delta_0\right)^{-1}\Delta_{\omega}'\Gamma^{-1\prime} \end{matrix}\right. \\
&\qquad \left. \begin{matrix} -\Gamma^{-1}\left\{\mathbb{E}\left[\frac{G_1}{G}\psi t'\right] - E_{\omega}F_{\omega}^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right] - \Delta_{\omega}\right\}\mathcal{I}\left(\delta_0\right)^{-1} \\ \mathcal{I}\left(\delta_0\right)^{-1} \end{matrix} \right) \\
&= \begin{pmatrix} \begin{pmatrix} \Gamma^{-1}\left\{\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_{\omega}F_{\omega}^{-1}E_0' - \Delta_{\omega}\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\right\}\Gamma^{-1\prime} \\ -\Gamma^{-1}\left\{E_0 - E_{\omega}F_{\omega}^{-1}F_0 - \Delta_{\omega}\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\right\}F_{\omega}^{-1}E_{\omega}'\Gamma^{-1\prime} \\ -\Gamma^{-1}\left\{\mathbb{E}\left[\frac{G_1}{G}\psi t'\right] - E_{\omega}F_{\omega}^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right] - \Delta_{\omega}\right\}\mathcal{I}\left(\delta_0\right)^{-1}\Delta_{\omega}'\Gamma^{-1\prime} \end{pmatrix} & 0 \\ 0 & \mathcal{I}\left(\delta_0\right)^{-1} \end{pmatrix}.
\end{aligned}
$$

The upper-left-hand block of this matrix may be manipulated further as follows:

$$
\begin{aligned}
&\Gamma^{-1}\left\{\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_0'-\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\right\}\Gamma^{-1\prime}\\
&-\Gamma^{-1}\left\{E_0-E_\omega F_\omega^{-1}F_0-\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\right\}F_\omega^{-1}E_\omega'\Gamma^{-1\prime}\\
&-\Gamma^{-1}\left\{\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]-E_\omega F_\omega^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]-\Delta_\omega\right\}\mathcal{I}\left(\delta_0\right)^{-1}\Delta_\omega'\Gamma^{-1\prime}\\
&=\Gamma^{-1}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_0F_0^{-1}E_0'\right)\Gamma^{-1\prime}\\
&+\Gamma^{-1}\left(E_0F_0^{-1}E_0'-E_\omega F_\omega^{-1}E_0'-E_0F_\omega^{-1}E_\omega'+E_\omega F_\omega^{-1}F_0F_\omega^{-1}E_\omega'\right.\\
&\left.-\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]+\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]F_\omega^{-1}E_\omega'\right)\Gamma^{-1\prime}\\
&=\Gamma^{-1}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_0F_0^{-1}E_0'\right)\Gamma^{-1\prime}\\
&+\Gamma^{-1}\left(E_0F_0^{-1}E_0'-E_\omega F_\omega^{-1}E_0'-E_0F_\omega^{-1}E_\omega'+E_\omega F_\omega^{-1}F_0F_\omega^{-1}E_\omega'\right.\\
&\left.-\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}\left\{\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]-\mathbb{E}\left[\frac{G_1}{G}tt'\right]F_\omega^{-1}E_\omega'\right\}\right)\Gamma^{-1\prime}\\
&=\Gamma^{-1}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_0F_0^{-1}E_0'\right)\Gamma^{-1\prime}\\
&+\Gamma^{-1}\left(E_0F_0^{-1}E_0'-E_\omega F_\omega^{-1}E_0'-E_0F_\omega^{-1}E_\omega'+E_\omega F_\omega^{-1}F_0F_\omega^{-1}E_\omega'\right.\\
&\left.-\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}\Delta_\omega'\right)\Gamma^{-1\prime}\\
&=\Gamma^{-1}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_0F_0^{-1}E_0'\right)\Gamma^{-1\prime}\\
&+\Gamma^{-1}\left(E_0F_0^{-1}-E_\omega F_\omega^{-1}\right)F_0\left(E_0F_0^{-1}-E_\omega F_\omega^{-1}\right)'\Gamma^{-1\prime}\\
&-\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}\Delta_\omega'\Gamma^{-1\prime}.
\end{aligned}
$$

Detailed calculations for proof of Theorem 3.1

Details of $\widehat{\gamma}_{IPT}$ stochastic expansion Equation (39) in the main appendix follows from the fact that

$$
\begin{aligned}
\mathbb{E}\left[A_i\phi_i\right] &= -\mathbb{E}\left[\begin{pmatrix} \frac{D_i}{G_i}\frac{\partial\psi_i}{\partial\gamma'} - \Gamma & -\frac{D_i}{G_i}\frac{G_{1i}}{G_i}\psi_i t_i' + \mathbb{E}\left[\frac{G_1}{G}\psi t'\right] \\ 0 & -\frac{D_i}{G_i}\frac{G_{1i}}{G_i}t_i t_i' + \mathbb{E}\left[\frac{G_1}{G}tt'\right] \end{pmatrix}\right. \\
&\quad \left.\times \begin{pmatrix} \Gamma^{-1} & -\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \\ 0 & -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \end{pmatrix} \times \begin{pmatrix} \frac{D_i}{G_i}\psi_i \\ \left(\frac{D_i}{G_i}-1\right)t_i \end{pmatrix}\right] \\
&= -\mathbb{E}\left[\begin{pmatrix} \frac{D_i}{G_i}\frac{\partial\psi_i}{\partial\gamma'}\Gamma^{-1} - I_K & -\left(\frac{D_i}{G_i}\frac{\partial\psi_i}{\partial\gamma'}\Gamma^{-1} - I_K\right)\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} + \frac{D_i}{G_i}\frac{G_{1i}}{G_i}\psi_i t_i'\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} - \mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \\ 0 & \frac{D_i}{G_i}\frac{G_{1i}}{G_i}t_i t_i'\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} - I_{1+M} \end{pmatrix}\right. \\
&\quad \left.\times \begin{pmatrix} \frac{D_i}{G_i}\psi_i \\ \left(\frac{D_i}{G_i}-1\right)t_i \end{pmatrix}\right] \\
&= -\mathbb{E}\left[\begin{matrix} \left\{ \begin{matrix} \frac{D_i}{G_i}\frac{\partial\psi_i}{\partial\gamma'}\Gamma^{-1}\frac{D_i}{G_i}\psi_i - \frac{D_i}{G_i}\psi_i - \left(\frac{D_i}{G_i}\frac{\partial\psi_i}{\partial\gamma'}\Gamma^{-1} - I_K\right)\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\left(\frac{D_i}{G_i}-1\right)t_i \\ +\frac{D_i}{G_i}\frac{G_{1i}}{G_i}\psi_i t_i'\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\left(\frac{D_i}{G_i}-1\right)t_i - \mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\left(\frac{D_i}{G_i}-1\right)t_i \end{matrix} \right\} \\ \left(\frac{D_i}{G_i}\frac{G_{1i}}{G_i}t_i t_i'\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} - I_{1+M}\right)\left(\frac{D_i}{G_i}-1\right)t_i \end{matrix}\right] \\
&= -\left[\begin{matrix} \mathbb{E}\left[\frac{D}{G}\frac{\partial\psi}{\partial\gamma'}\Gamma^{-1}\frac{D}{G}\psi\right] - \mathbb{E}\left[\frac{D}{G}\frac{\partial\psi}{\partial\gamma'}\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\left(\frac{D}{G}-1\right)t\right] + \mathbb{E}\left[\frac{D}{G}\frac{G_1}{G}\psi t'\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\left(\frac{D}{G}-1\right)t\right] \\ \mathbb{E}\left[\frac{D}{G}\frac{G_1}{G}tt'\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\left(\frac{D}{G}-1\right)t\right] \end{matrix}\right] \\
&= -\left[\begin{matrix} \mathbb{E}\left[\frac{\partial\psi}{\partial\gamma'}\Gamma^{-1}\frac{1}{G}\psi\right] - \mathbb{E}\left[\frac{1-G}{G}\frac{\partial\psi}{\partial\gamma'}\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}t\right] + \mathbb{E}\left[\frac{1-G}{G}\frac{G_1}{G}\psi t'\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}t\right] \\ \mathbb{E}\left[\frac{1-G}{G}\frac{G_1}{G}tt'\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}t\right] \end{matrix}\right].
\end{aligned}
$$

Equation (41) in the main appendix follows from the fact that

$$
\begin{aligned}
\mathbb{E}\left[\phi_i\phi_i'\right] &= \begin{bmatrix} \Gamma^{-1} & -\Gamma^{-1}\Pi_* \\ 0 & -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \end{bmatrix} \\
&\quad \times \mathbb{E}\left[\begin{pmatrix} \frac{D_i}{G_i}\psi_i \\ \left(\frac{D_i}{G_i}-1\right)t_i \end{pmatrix}\begin{pmatrix} \frac{D_i}{G_i}\psi_i \\ \left(\frac{D_i}{G_i}-1\right)t_i \end{pmatrix}'\right] \times \begin{bmatrix} \Gamma^{-1} & -\Gamma^{-1}\Pi_* \\ 0 & -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \end{bmatrix}' \\
&= \begin{bmatrix} \Gamma^{-1} & -\Gamma^{-1}\Pi_* \\ 0 & -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \end{bmatrix} \times \mathbb{E}\begin{bmatrix} \frac{1}{G}\psi\psi' & \frac{1-G}{G}\psi t' \\ \frac{1-G}{G}t\psi' & \frac{1-G}{G}tt' \end{bmatrix} \times \begin{bmatrix} \Gamma^{-1\prime} & 0 \\ -\Pi_*'\Gamma^{-1\prime} & -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \Gamma^{-1}\mathbb{E}\left[\frac{1}{G}\psi\psi'\right] - \Gamma^{-1}\Pi_*\mathbb{E}\left[\frac{1-G}{G}t\psi'\right] & \Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}\psi t'\right] - \Gamma^{-1}\Pi_*\mathbb{E}\left[\frac{1-G}{G}tt'\right] \\ -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{1-G}{G}t\psi'\right] & -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{1-G}{G}tt'\right] \end{bmatrix} \\
&\quad \times \begin{bmatrix} \Gamma^{-1\prime} & 0 \\ -\Pi_*'\Gamma^{-1\prime} & -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \end{bmatrix} \\
&= \left[\begin{matrix} \Gamma^{-1}\mathbb{E}\left[\frac{1}{G}\psi\psi'\right]\Gamma^{-1\prime} - \Gamma^{-1}\Pi_*\mathbb{E}\left[\frac{1-G}{G}t\psi'\right]\Gamma^{-1\prime} - \Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}\psi t'\right]\Pi_*'\Gamma^{-1\prime} + \Gamma^{-1}\Pi_*\mathbb{E}\left[\frac{1-G}{G}tt'\right]\Pi_*'\Gamma^{-1\prime} \\ -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{1-G}{G}t\psi'\right]\Gamma^{-1\prime} + \mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{1-G}{G}tt'\right]\Pi_*'\Gamma^{-1\prime} \end{matrix}\right. \\
&\quad \left.\begin{matrix} -\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} + \Gamma^{-1}\Pi_*\mathbb{E}\left[\frac{1-G}{G}tt'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \\ \mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{1-G}{G}tt'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \end{matrix}\right].
\end{aligned}
$$

Rearranging the two off-diagonal blocks in the expression above yields

$$\begin{aligned}
& -\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}+\Gamma^{-1}\Pi_*\mathbb{E}\left[\frac{1-G}{G}tt'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \\
&= -\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}\left\{\psi-\Pi_* t\right\}t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1},
\end{aligned}$$

which is mean zero if $\Pi_* = \Pi_0$. The upper-right-hand block can be rearranged as follows:

$$\begin{aligned}
& \Gamma^{-1}\mathbb{E}\left[\frac{1}{G}\psi\psi'\right]\Gamma^{-1\prime}-\Gamma^{-1}\Pi_*\mathbb{E}\left[\frac{1-G}{G}t\psi'\right]\Gamma^{-1\prime}-\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}\psi t'\right]\Pi_*'\Gamma^{-1\prime}+\Gamma^{-1}\Pi_*\mathbb{E}\left[\frac{1-G}{G}tt'\right]\Pi_*'\Gamma^{-1\prime} \\
&= \Gamma^{-1}\left\{\mathbb{E}\left[\frac{1}{G}\psi\psi'\right]-\mathbb{E}\left[\frac{1-G}{G}qq'\right]\right\}\Gamma^{-1\prime}+\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}qq'\right]\Gamma^{-1\prime} \\
& -\Gamma^{-1}\Pi_*\mathbb{E}\left[\frac{1-G}{G}t\psi'\right]\Gamma^{-1\prime}-\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}\psi t'\right]\Pi_*'\Gamma^{-1\prime}+\Gamma^{-1}\Pi_*\mathbb{E}\left[\frac{1-G}{G}tt'\right]\Pi_*'\Gamma^{-1\prime} \\
&= \mathcal{I}(\gamma_0)^{-1} \\
& +\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}qq'\right]\Gamma^{-1\prime} \\
& -\Gamma^{-1}\Pi_*\mathbb{E}\left[\frac{1-G}{G}tq'\right]\Gamma^{-1\prime}-\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}qt'\right]\Pi_*'\Gamma^{-1\prime}+\Gamma^{-1}\Pi_*\mathbb{E}\left[\frac{1-G}{G}tt'\right]\Pi_*'\Gamma^{-1\prime} \\
&= \mathcal{I}(\gamma_0)^{-1}+\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}(q-\Pi_* t)(q-\Pi_* t)'\right]\Gamma^{-1\prime}.
\end{aligned}$$

The this equality follows from the fact that $\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}\psi t'\right]\Pi_*'\Gamma^{-1\prime}=\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}qt'\right]\Pi_*'\Gamma^{-1\prime}$. The second term to the right of the last equality is identically equal to zero if $q=\Pi_* t=\Pi_0 t$. Putting all these results together yields (41) in the main appendix after making use of the equalities $q=\Pi_* t=\Pi_0 t$.

Using (42) and (41) we get, for $q=1,...,K$,

$$B_q\mathbb{E}\left[\phi_i\phi_i'\right]=\left[\begin{array}{cc}
\left(\begin{array}{c}\mathbb{E}\left[\frac{\partial^2\psi}{\partial\gamma_q\partial\gamma'}\right]\left\{\mathcal{I}(\gamma_0)^{-1}+\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}(q-\Pi_* t)(q-\Pi_* t)'\right]\Gamma^{-1\prime}\right\} \\ +\mathbb{E}\left[\frac{G_1}{G}\frac{\partial\psi}{\partial\gamma_q}t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{1-G}{G}t\left\{\psi-\Pi_* t\right\}'\right]\Gamma^{-1\prime}\end{array}\right) \\
0 \\
-\mathbb{E}\left[\frac{\partial^2\psi}{\partial\gamma_q\partial\gamma'}\right]\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}\left\{\psi-\Pi_* t\right\}t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}-\mathbb{E}\left[\frac{G_1}{G}\frac{\partial\psi}{\partial\gamma_q}t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{1-G}{G}tt'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \\
0
\end{array}\right].$$

Using (43) and (41) we get, for $q=K+1,..,K+1+M$,

$$B_q\mathbb{E}\left[\phi_i\phi_i'\right]=\left(\begin{array}{c}
\left(\begin{array}{c}-\mathbb{E}\left[\frac{G_1}{G}t_{q-K}\frac{\partial\psi}{\partial\gamma'}\right]\left\{\mathcal{I}(\gamma_0)^{-1}+\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}(q-\Pi_* t)(q-\Pi_* t)'\right]\Gamma^{-1\prime}\right\} \\ -\mathbb{E}\left[\left(\frac{2G_1^2}{G^2}-\frac{G_2}{G}\right)t_{q-K}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{1-G}{G}t\left\{\psi-\Pi_* t\right\}'\right]\Gamma^{-1\prime} \\ -\mathbb{E}\left[\left(\frac{2G_1^2}{G^2}-\frac{G_2}{G}\right)t_{q-K}tt'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{1-G}{G}t\left\{\psi-\Pi_* t\right\}'\right]\Gamma^{-1\prime}\end{array}\right) \\
\left(\begin{array}{c}\mathbb{E}\left[\frac{G_1}{G}t_{q-K}\frac{\partial\psi}{\partial\gamma'}\right]\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}\left\{\psi-\Pi_* t\right\}t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \\ +\mathbb{E}\left[\left(\frac{2G_1^2}{G^2}-\frac{G_2}{G}\right)t_{q-K}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{1-G}{G}tt'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\end{array}\right) \\
\mathbb{E}\left[\left(\frac{2G_1^2}{G^2}-\frac{G_2}{G}\right)t_{q-K}tt'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{1-G}{G}tt'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}
\end{array}\right).$$

Using these results we can show that (44) in the main appendix follows from the fact that

$$
\begin{aligned}
&-\frac{B^{-1}}{2}\sum_{q=1}^{T} B_q \mathbb{E}\left[\phi_i \phi_i'\right] e_q \\
&= -\frac{1}{2}\sum_{k=1}^{K}\left\{\begin{array}{c}\left(\begin{array}{c}\Gamma^{-1}\mathbb{E}\left[\frac{\partial^2\psi}{\partial\gamma_k\partial\gamma'}\right]\left\{\mathcal{I}(\gamma_0)^{-1}+\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}(q-\Pi_* t)(q-\Pi_* t)'\right]\Gamma^{-1\prime}\right\} \\ +\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\frac{\partial\psi}{\partial\gamma_k}t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{1-G}{G}t\{\psi-\Pi_* t\}'\right]\Gamma^{-1\prime}\end{array}\right) \\ 0\end{array}\right\} e_k \\
&-\frac{1}{2}\sum_{q=1}^{1+M}\left\{\begin{array}{c}\left(\begin{array}{c}\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}t_q\frac{\partial\psi}{\partial\gamma'}\right]\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}\{\psi-\Pi_* t\}t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \\ +\Gamma^{-1}\mathbb{E}\left[\left(\frac{2G_1^2}{G^2}-\frac{G_2}{G}\right)t_q\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{1-G}{G}tt'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \\ -\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\left(\frac{2G_1^2}{G^2}-\frac{G_2}{G}\right)t_q tt'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{1-G}{G}tt'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\end{array}\right) \\ -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\left(\frac{2G_1^2}{G^2}-\frac{G_2}{G}\right)t_q tt'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{1-G}{G}tt'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\end{array}\right\} e_q \\
&= -\frac{1}{2}\sum_{k=1}^{K}\left\{\begin{array}{c}\left(\begin{array}{c}\Gamma^{-1}\mathbb{E}\left[\frac{\partial^2\psi}{\partial\gamma_k\partial\gamma'}\right]\left\{\mathcal{I}(\gamma_0)^{-1}+\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}(q-\Pi_* t)(q-\Pi_* t)'\right]\Gamma^{-1\prime}\right\} \\ +\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\frac{\partial\psi}{\partial\gamma_k}t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{1-G}{G}t\{\psi-\Pi_* t\}'\right]\Gamma^{-1\prime}\end{array}\right) \\ 0\end{array}\right\} e_k \\
&-\frac{1}{2}\sum_{q=1}^{1+M}\left\{\begin{array}{c}\left(\begin{array}{c}\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}t_q\frac{\partial\psi}{\partial\gamma'}\right]\Gamma^{-1}\mathbb{E}\left[\frac{1-G}{G}\{\psi-\Pi_* t\}t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1} \\ -\Gamma^{-1}\mathbb{E}\left[\left(\frac{2G_1^2}{G^2}-\frac{G_2}{G}\right)t_q\{q-\Pi_* t\}t'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{1-G}{G}tt'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\end{array}\right) \\ -\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\left(\frac{2G_1^2}{G^2}-\frac{G_2}{G}\right)t_q tt'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\mathbb{E}\left[\frac{1-G}{G}tt'\right]\mathbb{E}\left[\frac{G_1}{G}tt'\right]^{-1}\end{array}\right\} e_q.
\end{aligned}
$$

The first K rows of which equal (44) after making additional simplifications due to the equality $q = \Pi_* t = \Pi_0 t$.

Details of $\widehat{\gamma}_{AIPW}$ stochastic expansion Equation (72) in the main appendix can be derived as follows. From (95) we have

$$
\left(M'V^{-1}M\right)^{-1}M'V^{-1} = \left(\begin{array}{ccc}\Gamma^{-1} & -\Gamma^{-1}E_\omega F_\omega^{-1} & -\Gamma^{-1}\Delta_\omega\mathcal{I}(\delta_0)^{-1} \\ 0 & 0 & -\mathcal{I}(\delta_0)^{-1}\end{array}\right),
$$

while (94) implies that

$$
V^{-1}M = \left(\begin{array}{cc}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Gamma & -\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right] \\ -F_\omega^{-1}E_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Gamma & F_\omega^{-1}E_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right] \\ 0 & -I_{1+M}\end{array}\right).
$$

Multiplying out we get

$$
\begin{aligned}
& V^{-1} M \left(M' V^{-1} M\right)^{-1} M' V^{-1} \\
& = \begin{pmatrix}
\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1} \Gamma & -\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1} \mathbb{E}\left[\frac{G_1}{G}\psi t'\right] \\
-F_\omega^{-1} E_\omega' \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1} \Gamma & F_\omega^{-1} E_\omega' \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1} \mathbb{E}\left[\frac{G_1}{G}\psi t'\right] \\
0 & -I_{1+M}
\end{pmatrix} \\
& \times \begin{pmatrix}
\Gamma^{-1} & -\Gamma^{-1} E_\omega F_\omega^{-1} & -\Gamma^{-1} \Delta_\omega \mathcal{I}\left(\delta_0\right)^{-1} \\
0 & 0 & -\mathcal{I}\left(\delta_0\right)^{-1}
\end{pmatrix} \\
& = \left(\begin{matrix}
\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1} & -\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right) E_\omega F_\omega^{-1} \\
-F_\omega^{-1} E_\omega' \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1} & F_\omega^{-1} E_\omega' \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1} E_\omega F_\omega^{-1} \\
0 & 0
\end{matrix} \right. \\
& \qquad \left. \begin{matrix}
-\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right) \Delta_\omega \mathcal{I}\left(\delta_0\right)^{-1} + \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1} \Pi_S \\
F_\omega^{-1} E_\omega' \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1} \Delta_\omega \mathcal{I}\left(\delta_0\right)^{-1} - F_\omega^{-1} E_\omega' \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1} \Pi_S \\
\mathcal{I}\left(\delta_0\right)^{-1}
\end{matrix} \right),
\end{aligned}
$$

where we recall that $\Pi_S = \mathbb{E}\left[\frac{G_1}{G}\psi t'\right] \mathcal{I}\left(\delta_0\right)^{-1}$.

Now using the expression for V^{-1} given in (93) above we have

$$
\begin{aligned}
L &= V^{-1} - V^{-1}M\left(M'V^{-1}M\right)^{-1}M'V^{-1} \\
&= \begin{bmatrix}
\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1} & -\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}E_\omega F_\omega^{-1} & 0 \\
-F_\omega^{-1}E_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1} & \left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} & 0 \\
\mathcal{I}(\delta_0)^{-1}\,\mathbb{E}\left[\frac{G_1}{G}tt'\right]F_\omega^{-1}E_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1} & -\mathcal{I}(\delta_0)^{-1}\,\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} & \mathcal{I}(\delta_0)^{-1}
\end{bmatrix} \\
&\quad - \begin{pmatrix}
\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1} & -\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)E_\omega F_\omega^{-1} & -\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)\Delta_\omega\mathcal{I}(\delta_0)^{-1} + \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Pi_S \\
-F_\omega^{-1}E_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1} & F_\omega^{-1}E_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}E_\omega F_\omega^{-1} & F_\omega^{-1}E_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Delta_\omega\mathcal{I}(\delta_0)^{-1} - F_\omega^{-1}E_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Pi_S \\
0 & 0 & \mathcal{I}(\delta_0)^{-1}
\end{pmatrix} \\
&= \begin{pmatrix}
0 & 0 & \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)\Delta_\omega\mathcal{I}(\delta_0)^{-1} - \left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Pi_S \\
0 & \left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} - F_\omega^{-1}E_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}E_\omega F_\omega^{-1} & -F_\omega^{-1}E_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Delta_\omega\mathcal{I}(\delta_0)^{-1} + F_\omega^{-1}E_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Pi_S \\
\mathcal{I}(\delta_0)^{-1}\,\mathbb{E}\left[\frac{G_1}{G}tt'\right]F_\omega^{-1}E_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1} & -\mathcal{I}(\delta_0)^{-1}\,\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} & 0
\end{pmatrix}.
\end{aligned}
$$

We can simplify this expression further. Note that

$$
\begin{aligned}
\Delta_\omega\mathcal{I}(\delta_0)^{-1} &= \mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathcal{I}(\delta_0)^{-1} - E_\omega F_\omega^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\mathcal{I}(\delta_0)^{-1} \\
&= \Pi_S - E_\omega F_\omega^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\mathcal{I}(\delta_0)^{-1} \\
&= \Pi_S - \Pi_\omega\mathbb{E}\left[\frac{G_1}{G}tt'\right]\mathcal{I}(\delta_0)^{-1},
\end{aligned}
$$

where $\Pi_\omega = E_\omega F_\omega^{-1}$. Define

$$
\Pi_{S\omega} = \mathbb{E}\left[\frac{G_1}{G}\Pi_\omega tt'\right]\mathcal{I}(\delta_0)^{-1} = \mathbb{E}\left[\frac{D}{G}\Pi_\omega tS_\delta'\right]\mathcal{I}(\delta_0)^{-1}.
$$

and also note

$$\left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} = F_\omega^{-1} + F_\omega^{-1}E_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1} - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}E_\omega F_\omega^{-1}.$$
$$= F_\omega^{-1} + \Pi_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1} - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Pi_\omega.$$

Substituting into the expression for L given above yields

$$L = \begin{pmatrix} 0 & 0 & -\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Pi_{S\omega} \\ 0 & F_\omega^{-1} & \Pi_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Pi_{S\omega} \\ \Pi_{S\omega}'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)^{-1} & -\mathcal{I}\left(\delta_0\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} & 0 \end{pmatrix}.$$

Under Assumption 2.1 we have $\Delta_\omega = 0$, $\Pi_\omega = \Pi_0$, $\Pi_{S\omega} = \Pi_S$ and $\mathcal{I}\left(\gamma_0\right)^{-1} = \Gamma^{-1}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1}E_\omega'\right)\Gamma^{-1\prime}$. After imposing these equalities we get (72) of the main appendix.

Equation (73) follows from the fact that

$$-B^{-1}\mathbb{E}\left[A_i\phi_i\right] = -\begin{pmatrix} -\Upsilon & H \\ H' & L \end{pmatrix}\mathbb{E}\left[\begin{pmatrix} 0 & \left(M_i - M\right)' \\ \left(M_i - M\right) & \xi_i \end{pmatrix}\begin{pmatrix} -H \\ -L \end{pmatrix}m_i\right]$$
$$= -\mathbb{E}\left[\begin{pmatrix} H\left(M_i - M\right) & -\Upsilon\left(M_i - M\right)' + H\xi_i \\ L\left(M_i - M\right) & H'\left(M_i - M\right)' + L\xi_i \end{pmatrix}\begin{pmatrix} -H \\ -L \end{pmatrix}m_i\right]$$
$$= -\mathbb{E}\left[\begin{pmatrix} -H\left(M_i - M\right)H + \Upsilon\left(M_i - M\right)'L - H\xi_i L \\ -L\left(M_i - M\right)H - H'\left(M_i - M\right)'L - L\xi_i L \end{pmatrix}m_i\right]$$
$$= -\mathbb{E}\left[\begin{pmatrix} -HM_iH + \Upsilon M_i'L - H\xi_i L \\ -LM_iH - H'M_i'L - L\xi_i L \end{pmatrix}m_i\right]$$

where we use the fact that m_i is mean zero.

Equation (74) of the main appendix follows from the fact that, using (57) and (71),

$$\mathbb{E}\left[M_i H m_i\right] = \mathbb{E}\left[\begin{pmatrix} \frac{D}{G}\frac{\partial\psi(\beta)}{\partial\gamma'} & -\frac{D}{G}\frac{G_1}{G}\psi t' \\ 0 & -\frac{D}{G}\frac{G_1}{G}tt' \\ 0 & J \end{pmatrix}\right.$$

$$\left.\begin{pmatrix} \Gamma^{-1} & -\Gamma^{-1}E_\omega F_\omega^{-1} & -\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1} \\ 0 & 0 & -\mathcal{I}\left(\delta_0\right)^{-1} \end{pmatrix}\begin{pmatrix} \frac{D}{G}\psi \\ \left(\frac{D}{G}-1\right)t \\ \frac{D-G}{G(1-G)}G_1 t \end{pmatrix}\right]$$

$$= \mathbb{E}\left[\begin{pmatrix} \frac{D}{G}\frac{\partial\psi(\beta)}{\partial\gamma'}\Gamma^{-1} & -\frac{D}{G}\frac{\partial\psi(\beta)}{\partial\gamma'}\Gamma^{-1}\Pi_\omega & -\frac{D}{G}\frac{\partial\psi(\beta)}{\partial\gamma'}\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1} + \frac{D}{G}\frac{G_1}{G}\psi t'\mathcal{I}\left(\delta_0\right)^{-1} \\ 0 & 0 & \frac{D}{G}\frac{G_1}{G}tt'\mathcal{I}\left(\delta_0\right)^{-1} \\ 0 & 0 & -J\mathcal{I}\left(\delta_0\right)^{-1} \end{pmatrix}\begin{pmatrix} \frac{D}{G}\psi \\ \left(\frac{D}{G}-1\right)t \\ \frac{D-G}{G(1-G)}G_1 t \end{pmatrix}\right]$$

$$= \mathbb{E}\left[\begin{array}{c} \frac{D}{G^2}\frac{\partial\psi(\beta)}{\partial\gamma'}\Gamma^{-1}\psi - \frac{D}{G}\left(\frac{D}{G}-1\right)\frac{\partial\psi(\beta)}{\partial\gamma'}\Gamma^{-1}\Pi_\omega t - \frac{D}{G}\frac{\partial\psi(\beta)}{\partial\gamma'}\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}\frac{D-G}{G(1-G)}G_1 t + \frac{D}{G}\frac{G_1}{G}\psi t'\mathcal{I}\left(\delta_0\right)^{-1}\frac{D-G}{G(1-G)}G_1 t \\ \frac{D}{G}\frac{G_1}{G}tt'\mathcal{I}\left(\delta_0\right)^{-1}\frac{D-G}{G(1-G)}G_1 t \\ -J\mathcal{I}\left(\delta_0\right)^{-1}\frac{D-G}{G(1-G)}G_1 t \end{array}\right]$$

recalling that $\Pi_\omega = E_\omega F_\omega^{-1}$.

Using (71) in the main appendix we then get

$$H\mathbb{E}\left[M_i H m_i\right] = \begin{pmatrix} \Gamma^{-1} & -\Gamma^{-1}E_\omega F_\omega^{-1} & -\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1} \\ 0 & 0 & -\mathcal{I}\left(\delta_0\right)^{-1} \end{pmatrix}$$

$$\times\,\mathbb{E}\left[\begin{array}{c} \begin{pmatrix} \frac{D}{G^2}\frac{\partial\psi(\beta)}{\partial\gamma'}\Gamma^{-1}\psi - \frac{D}{G}\left(\frac{D}{G}-1\right)\frac{\partial\psi(\beta)}{\partial\gamma'}\Gamma^{-1}\Pi_\omega t \\ -\frac{D}{G}\frac{\partial\psi(\beta)}{\partial\gamma'}\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}\frac{D-G}{G(1-G)}G_1 t + \frac{D}{G}\frac{G_1}{G}\psi t'\mathcal{I}\left(\delta_0\right)^{-1}\frac{D-G}{G(1-G)}G_1 t \end{pmatrix} \\ \frac{D}{G}\frac{G_1}{G}tt'\mathcal{I}\left(\delta_0\right)^{-1}\frac{D-G}{G(1-G)}G_1 t \\ -J\mathcal{I}\left(\delta_0\right)^{-1}\frac{D-G}{G(1-G)}G_1 t \end{array}\right]$$

$$= \mathbb{E}\left[\begin{array}{c} \begin{pmatrix} \Gamma^{-1}\frac{D}{G^2}\frac{\partial\psi(\beta)}{\partial\gamma'}\Gamma^{-1}\psi - \Gamma^{-1}\frac{D}{G}\left(\frac{D}{G}-1\right)\frac{\partial\psi(\beta)}{\partial\gamma'}\Gamma^{-1}\Pi_\omega t \\ -\Gamma^{-1}\frac{D}{G}\frac{\partial\psi(\beta)}{\partial\gamma'}\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}\frac{D-G}{G(1-G)}G_1 t \\ +\Gamma^{-1}\frac{D}{G}\frac{G_1}{G}\psi t'\mathcal{I}\left(\delta_0\right)^{-1}\frac{D-G}{G(1-G)}G_1 t \\ -\Gamma^{-1}E_\omega F_\omega^{-1}\frac{D}{G}\frac{G_1}{G}tt'\mathcal{I}\left(\delta_0\right)^{-1}\frac{D-G}{G(1-G)}G_1 t \\ +\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}J\mathcal{I}\left(\delta_0\right)^{-1}\frac{D-G}{G(1-G)}G_1 t \end{pmatrix} \\ \mathcal{I}\left(\delta_0\right)^{-1}J\mathcal{I}\left(\delta_0\right)^{-1}\frac{D-G}{G(1-G)}G_1 t \end{array}\right].$$

Manipulating the first K rows of this expression we get

$$\begin{aligned}
&\mathbb{E}\left[\Gamma^{-1}\frac{D}{G^2}\frac{\partial\psi\left(\beta\right)}{\partial\gamma'}\Gamma^{-1}\psi-\Gamma^{-1}\frac{D}{G}\left(\frac{D}{G}-1\right)\frac{\partial\psi\left(\beta\right)}{\partial\gamma'}\Gamma^{-1}\Pi_\omega t\right]\\
&-\mathbb{E}\left[\Gamma^{-1}\frac{D}{G}\frac{\partial\psi\left(\beta\right)}{\partial\gamma'}\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}\frac{D-G}{G\left(1-G\right)}G_1 t\right]\\
&+\mathbb{E}\left[\Gamma^{-1}\frac{D}{G}\frac{G_1}{G}\psi t'\mathcal{I}\left(\delta_0\right)^{-1}\frac{D-G}{G\left(1-G\right)}G_1 t\right]\\
&-\mathbb{E}\left[\Gamma^{-1}E_\omega F_\omega^{-1}\frac{D}{G}\frac{G_1}{G}tt'\mathcal{I}\left(\delta_0\right)^{-1}\frac{D-G}{G\left(1-G\right)}G_1 t\right]\\
&+\mathbb{E}\left[\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}J\mathcal{I}\left(\delta_0\right)^{-1}\frac{D-G}{G\left(1-G\right)}G_1 t\right]\\
&=\mathbb{E}\left[\Gamma^{-1}\frac{1}{G}\frac{\partial\psi\left(\beta\right)}{\partial\gamma'}\Gamma^{-1}\left(\psi-\Pi_\omega t\right)+\Gamma^{-1}\frac{\partial\psi\left(\beta\right)}{\partial\gamma'}\Gamma^{-1}\Pi_\omega t\right]\\
&-\mathbb{E}\left[\Gamma^{-1}\frac{D}{G}\frac{\partial\psi\left(\beta\right)}{\partial\gamma'}\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}S_\delta\right]\\
&+\mathbb{E}\left[\Gamma^{-1}\frac{D}{G}\frac{G_1}{G}\psi t'\mathcal{I}\left(\delta_0\right)^{-1}S_\delta\right]\\
&-\mathbb{E}\left[\Gamma^{-1}\Pi_\omega\frac{D}{G}\frac{G_1}{G}tt'\mathcal{I}\left(\delta_0\right)^{-1}S_\delta\right]\\
&+\mathbb{E}\left[\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}J\mathcal{I}\left(\delta_0\right)^{-1}S_\delta\right]\\
&=\mathbb{E}\left[\Gamma^{-1}\frac{1}{G}\frac{\partial\psi\left(\beta\right)}{\partial\gamma'}\Gamma^{-1}\left(\psi-\Pi_\omega t\right)+\Gamma^{-1}\frac{\partial\psi\left(\beta\right)}{\partial\gamma'}\Gamma^{-1}\Pi_\omega t\right]\\
&-\mathbb{E}\left[\Gamma^{-1}\frac{D}{G}\frac{\partial\psi\left(\beta\right)}{\partial\gamma'}\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}S_\delta\right]\\
&+\mathbb{E}\left[\Gamma^{-1}\frac{D}{G}\frac{G_1}{G}\left(\psi-\Pi_\omega t\right)t'\mathcal{I}\left(\delta_0\right)^{-1}S_\delta\right]\\
&+\mathbb{E}\left[\Gamma^{-1}\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}J\mathcal{I}\left(\delta_0\right)^{-1}S_\delta\right]\\
&=\Gamma^{-1}\mathbb{E}\left[\frac{1}{G}\frac{\partial\psi\left(\beta\right)}{\partial\gamma'}\Gamma^{-1}\left(\psi-\Pi_\omega t-\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}S_\delta\right)\right]+\Gamma^{-1}\mathbb{E}\left[\frac{\partial\psi\left(\beta\right)}{\partial\gamma'}\Gamma^{-1}\Pi_\omega t\right]\\
&+\Gamma^{-1}\mathbb{E}\left[\left\{\frac{D}{G}\frac{G_1}{G}\left(\psi-\Pi_\omega t\right)t'+\Delta_\omega\mathcal{I}\left(\delta_0\right)^{-1}J\right\}\mathcal{I}\left(\delta_0\right)^{-1}S_\delta\right].
\end{aligned}$$

Observe that if Assumption 2.1 additionally holds we have $\Pi_\omega t = q$ and $\Delta_\omega = 0$ so that the above equation simplifies to

$$\Gamma^{-1}\mathbb{E}\left[\frac{1}{G}\frac{\partial\psi\left(\beta\right)}{\partial\gamma'}\Gamma^{-1}\left(\psi-\Pi_\omega t\right)\right]+\Gamma^{-1}\mathbb{E}\left[\frac{\partial\psi\left(\beta\right)}{\partial\gamma'}\Gamma^{-1}\Pi_\omega t\right].$$

Equation (75) of the main appendix follows from the fact that, using (57) and (72),

$$\mathbb{E}\left[M_i' L m_i\right] = \mathbb{E}\left[\begin{pmatrix} \frac{D}{G}\left(\frac{\partial\psi}{\partial\gamma'}\right)' & 0 & 0 \\ -\frac{D}{G}\frac{G_1}{G}t\psi' & -\frac{D}{G}\frac{G_1}{G}tt' & J \end{pmatrix}\right.$$

$$\begin{pmatrix} 0 & 0 \\ 0 & F_\omega^{-1} \\ \Pi_{S\omega}'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1} & -\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} \end{pmatrix}$$

$$\left.\begin{pmatrix} -\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1}\Pi_{S\omega} \\ \Pi_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1}\Pi_{S\omega} \\ 0 \end{pmatrix}\begin{pmatrix} \frac{D\psi}{G} \\ \left(\frac{D}{G}-1\right)t \\ \frac{D-G}{G(1-G)}G_1 t \end{pmatrix}\right]$$

$$= \mathbb{E}\left[\begin{pmatrix} 0 & 0 \\ J\Pi_{S\omega}'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1} & -\frac{D}{G}\frac{G_1}{G}tt'F_\omega^{-1} - J\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} \end{pmatrix}\right.$$

$$\begin{pmatrix} -\frac{D}{G}\left(\frac{\partial\psi}{\partial\gamma'}\right)'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1}\Pi_{S\omega} \\ \frac{D}{G}\frac{G_1}{G}t\psi'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1}\Pi_{S\omega} - \frac{D}{G}\frac{G_1}{G}tt'\Pi_\omega'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1}\Pi_{S\omega} \end{pmatrix}$$

$$\left.\begin{pmatrix} \frac{D\psi}{G} \\ \left(\frac{D}{G}-1\right)t \\ \frac{D-G}{G(1-G)}G_1 t \end{pmatrix}\right]$$

$$= \mathbb{E}\left[\begin{pmatrix} 0 & 0 \\ J\Pi_{S\omega}'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1} & -\frac{D}{G}\frac{G_1}{G}tt'F_\omega^{-1} - J\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} \end{pmatrix}\right.$$

$$\begin{pmatrix} -\frac{D}{G}\left(\frac{\partial\psi}{\partial\gamma'}\right)'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1}\Pi_{S\omega} \\ \frac{D}{G}\frac{G_1}{G}t\left(\psi - \Pi_\omega t\right)'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1}\Pi_{S\omega} \end{pmatrix}$$

$$\left.\begin{pmatrix} \frac{D\psi}{G} \\ \left(\frac{D}{G}-1\right)t \\ \frac{D-G}{G(1-G)}G_1 t \end{pmatrix}\right]$$

$$= \mathbb{E}\left[\begin{pmatrix} -\frac{D}{G}\left(\frac{\partial\psi}{\partial\gamma'}\right)'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1}\Pi_{S\omega}\frac{D-G}{G(1-G)}G_1 t \\ J\Pi_{S\omega}'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1}\frac{D\psi}{G} \\ -\frac{D}{G}\frac{G_1}{G}tt'F_\omega^{-1}\left(\frac{D}{G}-1\right)t - J\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}\left(\frac{D}{G}-1\right)t \\ \frac{D}{G}\frac{G_1}{G}t\left(\psi - \Pi_\omega t\right)'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - E_\omega F_\omega^{-1} E_\omega'\right)^{-1}\Pi_{S\omega}\frac{D-G}{G(1-G)}G_1 t \end{pmatrix}\right].$$

Using the definition of Υ given in (70) we then get the first K rows of $\Upsilon\mathbb{E}\left[M_i' L m_i\right]$ equal to

$$-\left(\Gamma^{-1}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)\Gamma^{-1\prime}+\Gamma^{-1}\Delta_\omega\Pi_S'\Gamma^{-1\prime}\right)\mathbb{E}\left[\frac{D}{G}\left(\frac{\partial\psi}{\partial\gamma'}\right)'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Pi_{S\omega}\frac{D-G}{G(1-G)}G_1 t\right]$$

$$+\Gamma^{-1}\Delta_\omega\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\left(\begin{array}{c} J\Pi_{S\omega}'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\frac{D\psi}{G} \\ -\frac{D}{G}\frac{G_1}{G}tt'F_\omega^{-1}\left(\frac{D}{G}-1\right)t-J\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega-E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}\left(\frac{D}{G}-1\right)t \\ \frac{D}{G}\frac{G_1}{G}t(\psi-\Pi_\omega t)'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E_\omega'\right)^{-1}\Pi_{S\omega}\frac{D-G}{G(1-G)}G_1 t \end{array}\right)\right].$$

If Assumption 2.1 also holds the above simplifies to

$$-\mathcal{I}(\gamma_0)^{-1}\mathbb{E}\left[\frac{D}{G}\left(\frac{\partial\psi}{\partial\gamma'}\right)'\Lambda^{-1}\Pi_S S_\delta\right].$$

Our derivation of (76) begins with the observation that

$$\begin{aligned}
H\mathbb{E}[\xi_i L m_i] &= H\mathbb{E}\left[\left(V_i - V + \sum_{q=1}^{K+1+M}\mathbb{E}\left[\frac{\partial V_i(\beta_0)}{\partial\beta_q}\right]e_q'\phi_i\right)Lm_i\right] \\
&= H\mathbb{E}[(V_i-V)Lm_i] + H\mathbb{E}\left[\sum_{q=1}^{K+1+M}\mathbb{E}\left[\frac{\partial V_i(\beta_0)}{\partial\beta_q}\right]e_q'\phi_i L m_i\right] \\
&= H\mathbb{E}[V_i L m_i] - \sum_{q=1}^{K+1+M}H\mathbb{E}\left[\frac{\partial V_i(\beta_0)}{\partial\beta_q}\right]\mathbb{E}\left[e_q'\left(\begin{array}{c} H \\ L \end{array}\right)m_i L m_i\right] \\
&= H\mathbb{E}[V_i L m_i] - \sum_{q=1}^{K+1+M}H\mathbb{E}\left[\frac{\partial V_i(\beta_0)}{\partial\beta_q}\right]\mathbb{E}\left[e_q' H m_i L m_i\right]
\end{aligned}$$

First consider the second term in the last line above. Note that

$$\begin{aligned}
\mathbb{E}\left[e_q' H m_i L m_i\right] &= \mathbb{E}\left[m_i' H' e_q L m_i\right] \\
&= L\mathbb{E}\left[m_i m_i'\right]H' e_q \\
&= LVHe_q \\
&= \left(V^{-1}-V^{-1}M\left(M'V^{-1}M\right)^{-1}M'V^{-1}\right)VV^{-1}M\left(M'V^{-1}M\right)^{-1} \\
&= \left(M'V^{-1}M\right)^{-1}M'V^{-1} - V^{-1}M\left(M'V^{-1}M\right)^{-1}M'V^{-1}M\left(M'V^{-1}M\right)^{-1} \\
&= \left(M'V^{-1}M\right)^{-1}M'V^{-1} - V^{-1}M\left(M'V^{-1}M\right)^{-1} = 0.
\end{aligned}$$

Now consider the second term, $H\mathbb{E}[V_iLm_i]$. We start with the calculation, using (57), (58) and (72),

$$\mathbb{E}[V_iLm_i]$$

$$= \mathbb{E}\left[\left(\begin{array}{ccc} \frac{D}{G^2}\psi\psi' & \frac{D}{G}\omega\psi t' & 0 \\ \frac{D}{G}\omega t\psi' & \nu\omega tt' & 0 \\ 0 & \frac{D}{G}\frac{G_1}{G}tt' & -J \end{array}\right)\right.$$

$$\times\left(\begin{array}{cc} 0 & 0 \\ 0 & F_\omega^{-1} \\ \Pi'_{S\omega}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1} & -\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega-E'_\omega\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} \end{array}\right.$$

$$\left.\left.\begin{array}{c} -\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\Pi_{S\omega} \\ \Pi'_\omega\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\Pi_{S\omega} \\ 0 \end{array}\right)\left(\begin{array}{c} \frac{D\psi}{G} \\ \left(\frac{D}{G}-1\right)t \\ \frac{D-G}{G(1-G)}G_1t \end{array}\right)\right]$$

$$= \mathbb{E}\left[\left(\begin{array}{cc} 0 & \frac{D}{G}\omega\psi t'F_\omega^{-1} \\ 0 & \nu\omega tt'F_\omega^{-1} \\ -J\Pi'_{S\omega}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1} & \frac{D}{G}\frac{G_1}{G}tt'F_\omega^{-1}+J\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega-E'_\omega\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} \end{array}\right.\right.$$

$$\left.\left.\begin{array}{c} -\frac{D}{G^2}\psi\psi'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\Pi_{S\omega}+\frac{D}{G}\omega\psi t'\Pi'_\omega\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\Pi_{S\omega} \\ -\frac{D}{G}\omega t\psi'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\Pi_{S\omega}+\nu\omega tt'\Pi'_\omega\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\Pi_{S\omega} \\ \frac{D}{G}\frac{G_1}{G}tt'\Pi'_\omega\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\Pi'_{S\omega} \end{array}\right)\left(\begin{array}{c} \frac{D\psi}{G} \\ \left(\frac{D}{G}-1\right)t \\ \frac{D-G}{G(1-G)}G_1t \end{array}\right)\right]$$

$$= \mathbb{E}\left[\begin{array}{c} \left(\begin{array}{c} \frac{D}{G}\omega\psi t'F_\omega^{-1}\left(\frac{D}{G}-1\right)t \\ -\frac{D}{G^2}\psi\psi'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\Pi_{S\omega}\frac{D-G}{G(1-G)}G_1t+\frac{D}{G}\omega\psi t'\Pi'_\omega\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\Pi_{S\omega}\frac{D-G}{G(1-G)}G_1t \end{array}\right) \\ \left(\begin{array}{c} \nu\omega tt'F_\omega^{-1}\left(\frac{D}{G}-1\right)t \\ -\frac{D}{G}\omega t\psi'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\Pi_{S\omega}\frac{D-G}{G(1-G)}G_1t+\nu\omega tt'\Pi'_\omega\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\Pi_{S\omega}\frac{D-G}{G(1-G)}G_1t \end{array}\right) \\ \left(\begin{array}{c} -J\Pi'_{S\omega}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\frac{D\psi}{G} \\ \frac{D}{G}\frac{G_1}{G}tt'F_\omega^{-1}\left(\frac{D}{G}-1\right)t+J\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega-E'_\omega\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}\left(\frac{D}{G}-1\right)t \\ \frac{D}{G}\frac{G_1}{G}tt'\Pi'_\omega\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\Pi'_{S\omega}\frac{D-G}{G(1-G)}G_1t \end{array}\right) \end{array}\right].$$

Using the expression for H given in (71) we get the first K rows of $H\mathbb{E}[V_iLm_i]$ equal to

$$\mathbb{E}\left[\Gamma^{-1}\left\{\left(\begin{array}{c} \frac{D}{G}\omega\psi t'F_\omega^{-1}\left(\frac{D}{G}-1\right)t \\ -\frac{D}{G^2}\psi\psi'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\Pi_{S\omega}\frac{D-G}{G(1-G)}G_1t+\frac{D}{G}\omega\psi t'\Pi'_\omega\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\Pi_{S\omega}\frac{D-G}{G(1-G)}G_1t \end{array}\right)\right.\right.$$

$$-\Gamma^{-1}E_\omega F_\omega^{-1}\left\{\left(\begin{array}{c} \nu\omega tt'F_\omega^{-1}\left(\frac{D}{G}-1\right)t \\ -\frac{D}{G}\omega t\psi'\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\Pi_{S\omega}\frac{D-G}{G(1-G)}G_1t+\nu\omega tt'\Pi'_\omega\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\Pi_{S\omega}\frac{D-G}{G(1-G)}G_1t \end{array}\right)\right\}$$

$$\left.\left.-\Gamma^{-1}\Delta_\omega\mathcal{I}(\delta_0)^{-1}\left\{\left(\begin{array}{c} -J\Pi'_{S\omega}\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\frac{D\psi}{G} \\ \frac{D}{G}\frac{G_1}{G}tt'F_\omega^{-1}\left(\frac{D}{G}-1\right)t+J\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega-E'_\omega\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}\left(\frac{D}{G}-1\right)t \\ \frac{D}{G}\frac{G_1}{G}tt'\Pi'_\omega\left(\mathbb{E}\left[\frac{\psi\psi'}{G}\right]-E_\omega F_\omega^{-1}E'_\omega\right)^{-1}\Pi'_{S\omega}\frac{D-G}{G(1-G)}G_1t \end{array}\right)\right\}\right\}\right].$$

Under Assumption 2.1 the final term is identically equal to zero and the remaining two terms simplify as follows:

$$
\begin{aligned}
&\Gamma^{-1}\mathbb{E}\left[\omega\left\{\frac{D}{G}\psi-\nu\Pi_0 t\right\}t'F_0^{-1}\left(\frac{D}{G}-1\right)t\right]\\
&-\Gamma^{-1}\mathbb{E}\left[\frac{D}{G^2}\psi\psi'\Lambda^{-1}\Pi_S S_\delta-\frac{D}{G}\omega\psi\left(\Pi_0 t_0\right)'\Lambda^{-1}\Pi_S S_\delta\right]\\
&+\Gamma^{-1}\Pi_0\mathbb{E}\left[\frac{D}{G}\omega t\psi'\Lambda^{-1}\Pi_S S_\delta-\nu\omega t\left(\Pi_0 t_0\right)'\Lambda^{-1}\Pi_S S_\delta\right]\\
&=\Gamma^{-1}\mathbb{E}\left[\omega\left\{\frac{D}{G}\psi-\nu\Pi_0 t\right\}t'F_0^{-1}\left(\frac{D}{G}-1\right)t\right]\\
&-\Gamma^{-1}\mathbb{E}\left[\frac{D}{G}\psi\left(\frac{D}{G}\psi-\omega\Pi_0 t_0\right)'\Lambda^{-1}\Pi_S S_\delta\right]\\
&+\Gamma^{-1}\mathbb{E}\left[\frac{D}{G}\omega\Pi_0 t\psi'\Lambda^{-1}\Pi_S S_\delta-\nu\omega\left(\Pi_0 t\right)\left(\Pi_0 t_0\right)'\Lambda^{-1}\Pi_S S_\delta\right]\\
&=\Gamma^{-1}\mathbb{E}\left[\omega\left\{\frac{D}{G}\psi-\nu\Pi_0 t\right\}t'F_0^{-1}\left(\frac{D}{G}-1\right)t\right]\\
&-\Gamma^{-1}\mathbb{E}\left[\frac{D}{G}\psi\left(\frac{D}{G}\psi-\omega\Pi_0 t_0\right)'\Lambda^{-1}\Pi_S S_\delta\right]\\
&+\Gamma^{-1}\mathbb{E}\left[\omega\left(\frac{D}{G}-\nu\right)\Pi_0 tt'\Pi_0\Lambda^{-1}\Pi_S S_\delta\right]\\
&=\Gamma^{-1}\Pi_0\mathbb{E}\left[\omega\left(\frac{D}{G}-\nu\right)\left(\frac{D}{G}-1\right)tt'F_0^{-1}t\right]\\
&-\Gamma^{-1}\mathbb{E}\left[\frac{D}{G}\psi\left(\frac{D}{G}\psi-\omega\Pi_0 t_0\right)'\Lambda^{-1}\Pi_S S_\delta\right]\\
&+\Gamma^{-1}\mathbb{E}\left[\omega\left(\frac{D}{G}-\nu\right)\Pi_0 tt'\Pi_0\Lambda^{-1}\Pi_S S_\delta\right],
\end{aligned}
$$

as given in (76) of the main appendix.

We now turn to the derivation of (81) in the main appendix. We begin by establishing some preliminary results. For $q=1,\ldots,K$ we have, using (57) and (58),

$$
\mathbb{E}\left[\frac{\partial M_i}{\partial\beta_q}\right]=\begin{pmatrix}\mathbb{E}\left[\frac{\partial^2\psi}{\partial\gamma_q\partial\gamma}\right] & -\mathbb{E}\left[\frac{G_1}{G}\frac{\partial\psi}{\partial\gamma_q}t'\right]\\ 0 & 0\\ 0 & 0\end{pmatrix},
$$

while for $q=K+1,\ldots,K+1+M$ we have

$$
\mathbb{E}\left[\frac{\partial M_i}{\partial\beta_q}\right]=\begin{pmatrix}\mathbb{E}\left[-\frac{G_1}{G}\frac{\partial\psi}{\partial\gamma'}t_{q-K}\right] & \mathbb{E}\left[\left(2\left(\frac{G_1}{G}\right)^2-\frac{G_2}{G}\right)\psi t't_{q-K}\right]\\ 0 & \mathbb{E}\left[\left(2\left(\frac{G_1}{G}\right)^2-\frac{G_2}{G}\right)tt't_{q-K}\right]\\ 0 & \mathbb{E}\left[\partial J_i\left(\delta\right)/\partial\delta_{q-K}\right]\end{pmatrix}.
$$

Using these results we can derive equation (81) as follows. Using (68), (80), (78) and (79) we get

$$
\begin{aligned}
& -\frac{B^{-1}}{2}\mathbb{E}\left[\sum\nolimits_{q=1}^{T}\phi_{q,i}B_q\phi_i\right] \\
&= -\frac{B^{-1}}{2}\sum\nolimits_{q=1}^{T}B_q\mathbb{E}\left[\phi_i\phi_i'\right]e_q \\
&= -\frac{B^{-1}}{2}\sum\nolimits_{q=1}^{K+1+M}B_q\begin{pmatrix} H\Omega H' \\ L\Omega H' \end{pmatrix}e_q - \frac{B^{-1}}{2}\sum\nolimits_{q=1}^{K+2(1+M)}B_{K+1+M+q}\begin{pmatrix} H\Omega L' \\ L+F\left(\Omega-V\right)F' \end{pmatrix}e_q \\
&= \frac{B^{-1}}{2}\sum\nolimits_{q=1}^{K+1+M}\begin{pmatrix} 0 & \mathbb{E}\left[\frac{\partial M_i'}{\partial\beta_q}\right] \\ \mathbb{E}\left[\frac{\partial M_i}{\partial\beta_q}\right] & 0 \end{pmatrix}\begin{pmatrix} H\Omega H' \\ L\Omega H' \end{pmatrix}e_q \\
&+ \frac{B^{-1}}{2}\sum\nolimits_{q=1}^{K+2(1+M)}\begin{pmatrix} \mathbb{E}\left[\frac{\partial^2 m_{q-K-1-M}(Z_i,\beta)}{\partial\beta\partial\beta'}\right] & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} H\Omega L' \\ L+L\left(\Omega-V\right)L' \end{pmatrix}e_q \\
&= \frac{B^{-1}}{2}\sum\nolimits_{q=1}^{K+1+M}\begin{pmatrix} \mathbb{E}\left[\frac{\partial M_i'}{\partial\beta_q}\right]L\Omega H' \\ \mathbb{E}\left[\frac{\partial M_i}{\partial\beta_q}\right]H\Omega H' \end{pmatrix}e_q \\
&+ \frac{B^{-1}}{2}\sum\nolimits_{q=1}^{K+2(1+M)}\begin{pmatrix} \mathbb{E}\left[\frac{\partial^2 m_{q-K-1-M}(Z_i,\beta)}{\partial\beta\partial\beta'}\right]H\Omega L' \\ 0 \end{pmatrix}e_q \\
&= -\frac{1}{2}\sum\nolimits_{q=1}^{K+1+M}\begin{pmatrix} -\Upsilon\mathbb{E}\left[\frac{\partial M_i'}{\partial\beta_q}\right]L\Omega H' + H\mathbb{E}\left[\frac{\partial M_i'}{\partial\beta_q}\right]H\Omega H' \\ H'\mathbb{E}\left[\frac{\partial M_i'}{\partial\beta_q}\right]L\Omega H' + L\mathbb{E}\left[\frac{\partial M_i'}{\partial\beta_q}\right]H\Omega H' \end{pmatrix}e_q \\
&- \frac{1}{2}\sum\nolimits_{q=1}^{K+2(1+M)}\begin{pmatrix} -\Upsilon\mathbb{E}\left[\frac{\partial^2 m_q(Z_i,\beta)}{\partial\beta\partial\beta'}\right]H\Omega L' \\ H'\mathbb{E}\left[\frac{\partial^2 m_q(Z_i,\beta)}{\partial\beta\partial\beta'}\right]H\Omega L' \end{pmatrix}e_q,
\end{aligned}
$$

which gives (81).

To calculate (82) we first compute, using (63), (71) and (72) and taking advantage of simplifications due to

Assumption 2.1 ,

$$
\begin{aligned}
L\Omega H' &= \begin{pmatrix} 0 & 0 & -\Lambda^{-1}\Pi_S \\ 0 & F_\omega^{-1} & \Pi_0'\Lambda^{-1}\Pi_S \\ \Pi_S'\Lambda^{-1} & -\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1} & 0 \end{pmatrix} \\
&\quad \times \begin{pmatrix} \mathbb{E}\left[\frac{\psi\psi'}{G}\right] & E_0 & \mathbb{E}\left[\frac{G_1}{G}\psi t'\right] \\ E_0' & F_0 & \mathbb{E}\left[\frac{G_1}{G}tt'\right] \\ \mathbb{E}\left[\frac{G_1}{G}t\psi'\right] & \mathbb{E}\left[\frac{G_1}{G}tt'\right] & \mathcal{I}(\delta_0) \end{pmatrix} \times \begin{pmatrix} \Gamma^{-1\prime} & 0 \\ -\Pi_0'\Gamma^{-1\prime} & 0 \\ 0 & -\mathcal{I}(\delta_0)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} -\Lambda^{-1}\Pi_S\mathbb{E}\left[\frac{G_1}{G}t\psi'\right] & -\Lambda^{-1}\Pi_S\mathbb{E}\left[\frac{G_1}{G}tt'\right] & -\Lambda^{-1}\Pi_S\mathcal{I}(\delta_0) \\ F_\omega^{-1}E_0' + \Pi_0'\Lambda^{-1}\Pi_S\mathbb{E}\left[\frac{G_1}{G}t\psi'\right] & F_\omega^{-1}F_0 + \Pi_0'\Lambda^{-1}\Pi_S\mathbb{E}\left[\frac{G_1}{G}tt'\right] & F_\omega^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right] + \Pi_0'\Lambda^{-1}\Pi_S\mathcal{I}(\delta_0) \\ \begin{pmatrix} \Pi_S'\Lambda^{-1}\mathbb{E}\left[\frac{\psi\psi'}{G}\right] \\ -\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}E_0' \end{pmatrix} & \begin{pmatrix} \Pi_S'\Lambda^{-1}E_0 \\ -\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}F_0 \end{pmatrix} & \begin{pmatrix} \Pi_S'\Lambda^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right] \\ -\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right] \end{pmatrix} \end{pmatrix} \\
&\quad \times \begin{pmatrix} \Gamma^{-1\prime} & 0 \\ -\Pi_0'\Gamma^{-1\prime} & 0 \\ 0 & -\mathcal{I}(\delta_0)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} -\Lambda^{-1}\Pi_S\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Gamma^{-1\prime} \\ +\Lambda^{-1}\Pi_S\mathbb{E}\left[\frac{G_1}{G}tt'\right]\Pi_0'\Gamma^{-1\prime} \end{pmatrix} & \Lambda^{-1}\Pi_S \\ \begin{pmatrix} F_\omega^{-1}E_0'\Gamma^{-1\prime} + \Pi_0'\Lambda^{-1}\Pi_S\mathbb{E}\left[\frac{G_1}{G}t\psi'\right]\Gamma^{-1\prime} \\ -F_\omega^{-1}F_0\Pi_0'\Gamma^{-1\prime} - \Pi_0'\Lambda^{-1}\Pi_S\mathbb{E}\left[\frac{G_1}{G}tt'\right]\Pi_0'\Gamma^{-1\prime} \end{pmatrix} & -F_\omega^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\mathcal{I}(\delta_0)^{-1} - \Pi_0'\Lambda^{-1}\Pi_S \\ \begin{pmatrix} \Pi_S'\Lambda^{-1}\mathbb{E}\left[\frac{\psi\psi'}{G}\right]\Gamma^{-1\prime} - \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}E_0'\Gamma^{-1\prime} \\ -\Pi_S'\Lambda^{-1}E_0\Pi_0'\Gamma^{-1\prime} + \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}F_0\Pi_0'\Gamma^{-1\prime} \end{pmatrix} & \begin{pmatrix} -\Pi_S'\Lambda^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right]\mathcal{I}(\delta_0)^{-1} \\ +\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\mathcal{I}(\delta_0)^{-1} \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \Lambda^{-1}\Pi_S \\ 0 & -F_\omega^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\mathcal{I}(\delta_0)^{-1} - \Pi_0'\Lambda^{-1}\Pi_S \\ \left(\Pi_S'\Lambda^{-1}\mathbb{E}\left[\frac{\psi\psi'}{G}\right]\Gamma^{-1\prime} - \Pi_S'\Lambda^{-1}E_0\Pi_0'\Gamma^{-1\prime}\right) & \begin{pmatrix} -\Pi_S'\Lambda^{-1}\Pi_S \\ +\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\mathcal{I}(\delta_0)^{-1} \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \Lambda^{-1}\Pi_S \\ 0 & -F_\omega^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\mathcal{I}(\delta_0)^{-1} - \Pi_0'\Lambda^{-1}\Pi_S \\ \Pi_S'\Gamma^{-1\prime} & \begin{pmatrix} -\Pi_S'\Lambda^{-1}\Pi_S \\ +\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\left(F_\omega - E_\omega'\mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1}E_\omega\right)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\mathcal{I}(\delta_0)^{-1} \end{pmatrix} \end{pmatrix}.
\end{aligned}
$$

We can get a further simplification by observing that, by Henderson and Searle (1981, Eq. 17),

$$\begin{aligned}
\left(F_\omega - E_\omega' \mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1} E_\omega\right)^{-1} &= F_\omega^{-1} - F_\omega^{-1} E_\omega' \left(-\mathbb{E}\left[\frac{\psi\psi'}{G}\right] + E_\omega F_\omega^{-1} E_\omega'\right)^{-1} E_\omega F_\omega^{-1} \\
&= F_\omega^{-1} + \Pi_0' \Lambda^{-1} \Pi_0.
\end{aligned}$$

This gives

$$\begin{aligned}
& -\Pi_S' \Lambda^{-1} \Pi_S + \mathcal{I}(\delta_0)^{-1} \mathbb{E}\left[\frac{G_1}{G} tt'\right] \left(F_\omega - E_\omega' \mathbb{E}\left[\frac{\psi\psi'}{G}\right]^{-1} E_\omega\right)^{-1} \mathbb{E}\left[\frac{G_1}{G} tt'\right] \mathcal{I}(\delta_0)^{-1} \\
&= -\Pi_S' \Lambda^{-1} \Pi_S + \mathcal{I}(\delta_0)^{-1} \mathbb{E}\left[\frac{G_1}{G} tt'\right] \left(F_\omega^{-1} + \Pi_0' \Lambda^{-1} \Pi_0\right) \mathbb{E}\left[\frac{G_1}{G} tt'\right] \mathcal{I}(\delta_0) \\
&= -\Pi_S' \Lambda^{-1} \Pi_S + \mathcal{I}(\delta_0)^{-1} \mathbb{E}\left[\frac{G_1}{G} tt'\right] F_\omega^{-1} \mathbb{E}\left[\frac{G_1}{G} tt'\right] \mathcal{I}(\delta_0)^{-1} \\
&+ \mathcal{I}(\delta_0)^{-1} \mathbb{E}\left[\frac{G_1}{G} tt' \Pi_0'\right] \Lambda^{-1} \mathbb{E}\left[\frac{G_1}{G} \Pi_0 tt'\right] \mathcal{I}(\delta_0)^{-1} \\
&= \mathcal{I}(\delta_0)^{-1} \mathbb{E}\left[\frac{G_1}{G} tt'\right] F_\omega^{-1} \mathbb{E}\left[\frac{G_1}{G} tt'\right] \mathcal{I}(\delta_0)^{-1}.
\end{aligned}$$

Similar arguments give

$$\begin{aligned}
& -F_\omega^{-1} \mathbb{E}\left[\frac{G_1}{G} tt'\right] \mathcal{I}(\delta_0)^{-1} - \Pi_0' \Lambda^{-1} \Pi_S \\
& -F_\omega^{-1} \mathbb{E}\left[\frac{G_1}{G} tt'\right] \mathcal{I}(\delta_0)^{-1} - \Pi_0' \Lambda^{-1} \mathbb{E}\left[\frac{G_1}{G} \psi t'\right] \mathcal{I}(\delta_0)^{-1} \\
& -\left(F_\omega^{-1} + \Pi_0' \Lambda^{-1} \Pi_0\right) \mathbb{E}\left[\frac{G_1}{G} tt'\right] \mathcal{I}(\delta_0)^{-1}
\end{aligned}$$

In the end we have

$$L\Omega H' = \begin{pmatrix} 0 & \Lambda^{-1}\Pi_S \\ 0 & -\left(F_\omega^{-1} + \Pi_0' \Lambda^{-1} \Pi_0\right) \mathbb{E}\left[\frac{G_1}{G} tt'\right] \mathcal{I}(\delta_0)^{-1} \\ \Pi_S' \Gamma^{-1\prime} & \mathcal{I}(\delta_0)^{-1} \mathbb{E}\left[\frac{G_1}{G} tt'\right] F_\omega^{-1} \mathbb{E}\left[\frac{G_1}{G} tt'\right] \mathcal{I}(\delta_0)^{-1} \end{pmatrix}.$$

Using this result, (70), and the expression for $\mathbb{E}\left[\frac{\partial M_i}{\partial \beta_q}\right]$ (for $q = 1, \ldots, K$) given above we can then derive equation

(82) by multiplying out (maintaining Assumption 2.1 throughout)

$$
\begin{aligned}
&\Upsilon \mathbb{E}\left[\frac{\partial M_i'}{\partial \beta_q}\right] L\Omega H' \\
&= \begin{pmatrix} \mathcal{I}(\gamma_0)^{-1} & 0 \\ \Pi_S'\Gamma^{-1\prime} & \mathcal{I}(\delta_0)^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{E}\left[\frac{\partial^2 \psi}{\partial \gamma_q \partial \gamma}\right]' & 0 & 0 \\ -\mathbb{E}\left[\frac{G_1}{G}\frac{\partial \psi}{\partial \gamma_q} t'\right]' & 0 & 0 \end{pmatrix} \\
&\times \begin{pmatrix} 0 & \Lambda^{-1}\Pi_S \\ 0 & -\left(F_\omega^{-1} + \Pi_0'\Lambda^{-1}\Pi_0\right)\mathbb{E}\left[\frac{G_1}{G}tt'\right]\mathcal{I}(\delta_0)^{-1} \\ \Pi_S'\Gamma^{-1\prime} & \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]F_\omega^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\mathcal{I}(\delta_0)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{I}(\gamma_0)^{-1}\mathbb{E}\left[\frac{\partial^2 \psi}{\partial \gamma_q \partial \gamma}\right]' & 0 & 0 \\ \Pi_S'\Gamma^{-1\prime}\mathbb{E}\left[\frac{\partial^2 \psi}{\partial \gamma_q \partial \gamma}\right]' - \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}\frac{\partial \psi}{\partial \gamma_q} t'\right]' & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & \Lambda^{-1}\Pi_S \\ 0 & -\left(F_\omega^{-1} + \Pi_0'\Lambda^{-1}\Pi_0\right)\mathbb{E}\left[\frac{G_1}{G}tt'\right]\mathcal{I}(\delta_0)^{-1} \\ \Pi_S'\Gamma^{-1\prime} & \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]F_\omega^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\mathcal{I}(\delta_0)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \mathcal{I}(\gamma_0)^{-1}\mathbb{E}\left[\frac{\partial^2 \psi}{\partial \gamma_q \partial \gamma}\right]'\Lambda^{-1}\Pi_S \\ 0 & \Pi_S'\Gamma^{-1\prime}\mathbb{E}\left[\frac{\partial^2 \psi}{\partial \gamma_q \partial \gamma}\right]'\Lambda^{-1}\Pi_S - \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}\frac{\partial \psi}{\partial \gamma_q} t'\right]'\Lambda^{-1}\Pi_S \end{pmatrix}.
\end{aligned}
$$

The negative of the first K rows of this matrix give (82).

Similar calculations give (84). Using the expression for $\mathbb{E}\left[\frac{\partial M_i}{\partial \beta_q}\right]$ (for $q = K+1, \ldots, K+1+M$) given above we get

$$
\begin{aligned}
&\Upsilon\mathbb{E}\left[\frac{\partial M_i'}{\partial\beta_q}\right]L\Omega H' \\
&= \begin{pmatrix} \mathcal{I}(\gamma_0)^{-1} & 0 \\ \Pi_S'\Gamma^{-1\prime} & \mathcal{I}(\delta_0)^{-1} \end{pmatrix} \\
&\times \begin{pmatrix} \mathbb{E}\left[-\frac{G_1}{G}\frac{\partial\psi}{\partial\gamma'}t_{q-K}\right]' & 0 & 0 \\ \mathbb{E}\left[\left(2\left(\frac{G_1}{G}\right)^2-\frac{G_2}{G}\right)\psi t't_{q-K}\right]' & \mathbb{E}\left[\left(2\left(\frac{G_1}{G}\right)^2-\frac{G_2}{G}\right)tt't_{q-K}\right]' & \mathbb{E}\left[\partial J_i(\delta)/\partial\delta_{q-K}\right]' \end{pmatrix} \\
&\times \begin{pmatrix} 0 & \Lambda^{-1}\Pi_S \\ 0 & -\left(F_\omega^{-1}+\Pi_0'\Lambda^{-1}\Pi_0\right)\mathbb{E}\left[\frac{G_1}{G}tt'\right]\mathcal{I}(\delta_0)^{-1} \\ \Pi_S'\Gamma^{-1\prime} & \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]F_\omega^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\mathcal{I}(\delta_0)^{-1} \end{pmatrix} \\
&= \left(\begin{matrix} -\mathcal{I}(\gamma_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}\frac{\partial\psi}{\partial\gamma'}t_{q-K}\right]' \\ -\Pi_S'\Gamma^{-1\prime}\mathbb{E}\left[-\frac{G_1}{G}\frac{\partial\psi}{\partial\gamma'}t_{q-K}\right]'+\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\left(2\left(\frac{G_1}{G}\right)^2-\frac{G_2}{G}\right)\psi t't_{q-K}\right]' \end{matrix}\right. \\
&\qquad \left.\begin{matrix} 0 & 0 \\ \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\left(2\left(\frac{G_1}{G}\right)^2-\frac{G_2}{G}\right)tt't_{q-K}\right]' & \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\partial J_i(\delta)/\partial\delta_{q-K}\right]' \end{matrix}\right) \\
&\times \begin{pmatrix} 0 & \Lambda^{-1}\Pi_S \\ 0 & -\left(F_\omega^{-1}+\Pi_0'\Lambda^{-1}\Pi_0\right)\mathbb{E}\left[\frac{G_1}{G}tt'\right]\mathcal{I}(\delta_0)^{-1} \\ \Pi_S'\Gamma^{-1\prime} & \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]F_\omega^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\mathcal{I}(\delta_0)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\mathcal{I}(\gamma_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}\frac{\partial\psi}{\partial\gamma'}t_{q-K}\right]'\Lambda^{-1}\Pi_S \\ \text{(Not Needed)} & \text{(Not Needed)} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\mathcal{I}(\gamma_0)^{-1}\mathbb{E}\left[\frac{D}{G}\frac{\partial\psi}{\partial\gamma'}S_{\delta,q-K}'\right]'\Lambda^{-1}\Pi_S \\ \text{(Not Needed)} & \text{(Not Needed)} \end{pmatrix},
\end{aligned}
$$

the negative of the first K rows of which give (84).

To calculate (83) of the main text we first need to evaluate, using (63) and (71) above, as well as simplifications due to Assumption 2.1,

$$
\begin{aligned}
& H\Omega H' \\
&= \begin{pmatrix} \Gamma^{-1} & -\Gamma^{-1}\Pi_0 & 0 \\ 0 & 0 & -\mathcal{I}(\delta_0)^{-1} \end{pmatrix} \\
&\times \begin{pmatrix} \mathbb{E}\left[\frac{\psi\psi'}{G}\right] & E_0 & \mathbb{E}\left[\frac{G_1}{G}\psi t'\right] \\ E_0' & F_0 & \mathbb{E}\left[\frac{G_1}{G}tt'\right] \\ \mathbb{E}\left[\frac{G_1}{G}t\psi'\right] & \mathbb{E}\left[\frac{G_1}{G}tt'\right] & \mathcal{I}(\delta_0) \end{pmatrix} \times \begin{pmatrix} \Gamma^{-1\prime} & 0 \\ \Pi_0\Gamma^{-1\prime} & 0 \\ 0 & -\mathcal{I}(\delta_0)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \Gamma^{-1}\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - \Gamma^{-1}\Pi_0 E_0' & \Gamma^{-1}E_0 - \Gamma^{-1}\Pi_0 F_0 & \Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\psi t'\right] - \Gamma^{-1}\Pi_0\mathbb{E}\left[\frac{G_1}{G}tt'\right] \\ -\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\psi'\right] & -\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right] & -I_{1+M} \end{pmatrix} \\
&\times \begin{pmatrix} \Gamma^{-1\prime} & 0 \\ -\Pi_0\Gamma^{-1\prime} & 0 \\ 0 & -\mathcal{I}(\delta_0)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \Gamma^{-1}\mathbb{E}\left[\frac{\psi\psi'}{G}\right] - \Gamma^{-1}\Pi_0 E_0' & 0 & 0 \\ -\Pi_S' & -\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right] & -I_{1+M} \end{pmatrix} \\
&\times \begin{pmatrix} \Gamma^{-1\prime} & 0 \\ -\Pi_0\Gamma^{-1\prime} & 0 \\ 0 & -\mathcal{I}(\delta_0)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \Gamma^{-1}\mathbb{E}\left[\frac{\psi\psi'}{G}\right]\Gamma^{-1\prime} - \Gamma^{-1}\Pi_0 E_0'\Gamma^{-1\prime} & 0 \\ -\Pi_S'\Gamma^{-1\prime} + \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}tt'\right]\Pi_0\Gamma^{-1\prime} & \mathcal{I}(\delta_0)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{I}(\gamma_0)^{-1} & 0 \\ 0 & \mathcal{I}(\delta_0)^{-1} \end{pmatrix}.
\end{aligned}
$$

Using this result we can get (83), using the expression for $\mathbb{E}\left[\frac{\partial M_i}{\partial \beta_q}\right]$ (for $q = 1, \ldots, K$) given above,

$$H\mathbb{E}\left[\frac{\partial M_i}{\partial \beta_q}\right] H\Omega H' = \begin{pmatrix} \Gamma^{-1} & -\Gamma^{-1}\Pi_0 & 0 \\ 0 & 0 & -\mathcal{I}(\delta_0)^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{E}\left[\frac{\partial^2 \psi}{\partial \gamma_q \partial \gamma}\right] & -\mathbb{E}\left[\frac{G_1}{G}\frac{\partial \psi}{\partial \gamma_q} t'\right] \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} \mathcal{I}(\gamma_0)^{-1} & 0 \\ 0 & \mathcal{I}(\delta_0)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \Gamma^{-1}\mathbb{E}\left[\frac{\partial^2 \psi}{\partial \gamma_q \partial \gamma}\right] & -\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\frac{\partial \psi}{\partial \gamma_q} t'\right] \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} \mathcal{I}(\gamma_0)^{-1} & 0 \\ 0 & \mathcal{I}(\delta_0)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \Gamma^{-1}\mathbb{E}\left[\frac{\partial^2 \psi}{\partial \gamma_q \partial \gamma}\right]\mathcal{I}(\gamma_0)^{-1} & -\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\frac{\partial \psi}{\partial \gamma_q} t'\right]\mathcal{I}(\delta_0)^{-1} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \Gamma^{-1}\mathbb{E}\left[\frac{\partial^2 \psi}{\partial \gamma_q \partial \gamma}\right]\mathcal{I}(\gamma_0)^{-1} & -\Gamma^{-1}\mathbb{E}\left[\frac{D}{G}\frac{\partial \psi}{\partial \gamma_q} S'_\delta\right]\mathcal{I}(\delta_0)^{-1} \\ 0 & 0 \end{pmatrix},$$

the first K rows of which equal (83).

Equation (85) follows similarly. Using the expression for $\mathbb{E}\left[\frac{\partial M_i}{\partial \beta_q}\right]$ (for $q = K+1, \ldots, K+1+M$) given above we have

$$H\mathbb{E}\left[\frac{\partial M_i}{\partial \beta_q}\right] H\Omega H' = \begin{pmatrix} \Gamma^{-1} & -\Gamma^{-1}\Pi_0 & 0 \\ 0 & 0 & -\mathcal{I}(\delta_0)^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{E}\left[-\frac{G_1}{G}\frac{\partial \psi}{\partial \gamma'} t_{q-K}\right] & \mathbb{E}\left[\left(2\left(\frac{G_1}{G}\right)^2 - \frac{G_2}{G}\right)\psi t' t_{q-K}\right] \\ 0 & \mathbb{E}\left[\left(2\left(\frac{G_1}{G}\right)^2 - \frac{G_2}{G}\right) t t' t_{q-K}\right] \\ 0 & \mathbb{E}\left[\partial J_i(\delta)/\partial \delta_{q-K}\right] \end{pmatrix}$$

$$\times \begin{pmatrix} \mathcal{I}(\gamma_0)^{-1} & 0 \\ 0 & \mathcal{I}(\delta_0)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} -\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\frac{\partial \psi}{\partial \gamma'} t_{q-K}\right] & \Gamma^{-1}\mathbb{E}\left[\left(2\left(\frac{G_1}{G}\right)^2 - \frac{G_2}{G}\right)\psi t' t_{q-K}\right] - \Gamma^{-1}\Pi_0\mathbb{E}\left[\left(2\left(\frac{G_1}{G}\right)^2 - \frac{G_2}{G}\right) t t' t_{q-K}\right] \\ 0 & -\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\partial J_i(\delta)/\partial \delta_{q-K}\right] \end{pmatrix}$$

$$\times \begin{pmatrix} \mathcal{I}(\gamma_0)^{-1} & 0 \\ 0 & \mathcal{I}(\delta_0)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} -\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\frac{\partial \psi}{\partial \gamma'} t_{q-K}\right] & 0 \\ 0 & -\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\partial J_i(\delta)/\partial \delta_{q-K}\right] \end{pmatrix}$$

$$\times \begin{pmatrix} \mathcal{I}(\gamma_0)^{-1} & 0 \\ 0 & \mathcal{I}(\delta_0)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} -\Gamma^{-1}\mathbb{E}\left[\frac{G_1}{G}\frac{\partial \psi}{\partial \gamma'} t_{q-K}\right]\mathcal{I}(\gamma_0)^{-1} & 0 \\ 0 & -\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\partial J_i(\delta)/\partial \delta_{q-K}\right]\mathcal{I}(\delta_0)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} -\Gamma^{-1}\mathbb{E}\left[\frac{D}{G}\frac{\partial \psi}{\partial \gamma'} S'_{\delta,q-K}\right]\mathcal{I}(\gamma_0)^{-1} & 0 \\ 0 & -\mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\partial J_i(\delta)/\partial \delta_{q-K}\right]\mathcal{I}(\delta_0)^{-1} \end{pmatrix},$$

the first K rows of which equal (85).

Equation (87) follows from the fact that for $q = 1, \ldots, K$

$$\mathbb{E}\left[\frac{\partial^2 m_q(Z_i,\beta)}{\partial\beta\partial\beta'}\right] = \begin{pmatrix} \mathbb{E}\left[\frac{\partial^2\psi_q}{\partial\gamma\partial\gamma'}\right] & -\mathbb{E}\left[\frac{G_1}{G}\frac{\partial\psi_q}{\partial\gamma}t'\right] \\ -\mathbb{E}\left[\frac{G_1}{G}t\left(\frac{\partial\psi_q}{\partial\gamma}\right)'\right] & \mathbb{E}\left[\left(2\left(\frac{G_1}{G}\right)^2 - \frac{G_2}{G}\right)tt'\psi_q\right] \end{pmatrix},$$

so that, using other results derived previously, we can multiply out

$$\begin{aligned}
&\Upsilon\mathbb{E}\left[\frac{\partial^2 m_q(Z_i,\beta)}{\partial\beta\partial\beta'}\right]H\Omega L' \\
&= \begin{pmatrix} \mathcal{I}(\gamma_0)^{-1} & 0 \\ \Pi_S'\Gamma^{-1\prime} & \mathcal{I}(\delta_0)^{-1} \end{pmatrix} \times \begin{pmatrix} \mathbb{E}\left[\frac{\partial^2\psi_q}{\partial\gamma\partial\gamma'}\right] & -\mathbb{E}\left[\frac{G_1}{G}\frac{\partial\psi_q}{\partial\gamma}t'\right] \\ -\mathbb{E}\left[\frac{G_1}{G}t\left(\frac{\partial\psi_q}{\partial\gamma}\right)'\right] & \mathbb{E}\left[\left(2\left(\frac{G_1}{G}\right)^2 - \frac{G_2}{G}\right)tt'\psi_q\right] \end{pmatrix} \\
&\times \begin{pmatrix} 0 & 0 & \Gamma^{-1}\Pi_S \\ \Pi_S'\Lambda^{-1} & -\Pi_S'\left(F_\omega^{-1} + \Lambda^{-1}\Pi_0\right) & \Pi_S' F_\omega^{-1}\Pi_S \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{I}(\gamma_0)^{-1}\mathbb{E}\left[\frac{\partial^2\psi_q}{\partial\gamma\partial\gamma'}\right] & -\mathcal{I}(\gamma_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}\frac{\partial\psi_q}{\partial\gamma}t'\right] \\ \Pi_S'\Gamma^{-1\prime}\mathbb{E}\left[\frac{\partial^2\psi_q}{\partial\gamma\partial\gamma'}\right] - \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}t\left(\frac{\partial\psi_q}{\partial\gamma}\right)'\right] & -\Pi_S'\Gamma^{-1\prime}\mathbb{E}\left[\frac{G_1}{G}\frac{\partial\psi_q}{\partial\gamma}t'\right] + \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\left(2\left(\frac{G_1}{G}\right)^2 - \frac{G_2}{G}\right)t\psi_q t'\right] \end{pmatrix} \\
&\times \begin{pmatrix} 0 & 0 & \Gamma^{-1}\Pi_S \\ \Pi_S'\Lambda^{-1} & -\Pi_S'\left(F_\omega^{-1} + \Lambda^{-1}\Pi_0\right) & \Pi_S' F_\omega^{-1}\Pi_S \end{pmatrix} \\
&= \left(\begin{matrix} -\mathcal{I}(\gamma_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}\frac{\partial\psi_q}{\partial\gamma}t'\right]\Pi_S'\Lambda^{-1} \\ \text{(Not Needed)} \end{matrix} \right. \\
&\quad \begin{matrix} \mathcal{I}(\gamma_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}\frac{\partial\psi_q}{\partial\gamma}t'\right]\Pi_S'\left(F_\omega^{-1} + \Lambda^{-1}\Pi_0\right) \\ \text{(Not Needed)} \end{matrix} \\
&\quad \left. \begin{matrix} \mathcal{I}(\gamma_0)^{-1}\mathbb{E}\left[\frac{\partial^2\psi_q}{\partial\gamma\partial\gamma'}\right]\Gamma^{-1}\Pi_S - \mathcal{I}(\gamma_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}\frac{\partial\psi_q}{\partial\gamma}t'\right]\Pi_S' F_\omega^{-1}\Pi_S \\ \text{(Not Needed)} \end{matrix} \right),
\end{aligned}$$

noting that

$$-\mathcal{I}(\gamma_0)^{-1}\mathbb{E}\left[\frac{G_1}{G}\frac{\partial\psi_q}{\partial\gamma}t'\right]\Pi_S'\Lambda^{-1} = -\mathcal{I}(\gamma_0)^{-1}\mathbb{E}\left[\frac{D}{G}\frac{\partial\psi_q}{\partial\gamma}S_\delta'\right]\Pi_S'\Lambda^{-1}$$

we then get (87) as claimed.

We calculate (88) similarly. First note that, for $q = K+1, \ldots, K+1+M$, we have

$$\mathbb{E}\left[\frac{\partial^2 m_q(Z_i,\beta)}{\partial\beta\partial\beta'}\right] = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{E}\left[\left(2\left(\frac{G_1}{G}\right)^2 - \frac{G_2}{G}\right)tt't_q\right] \end{pmatrix}.$$

Multiplying out we then get, for $q = K+1, \ldots, K+1+M$,

$$
\begin{aligned}
&\Upsilon \mathbb{E}\left[\frac{\partial^2 m_q(Z_i,\beta)}{\partial\beta\partial\beta'}\right] H\Omega L' \\
&= \begin{pmatrix} \mathcal{I}(\gamma_0)^{-1} & 0 \\ \Pi_S'\Gamma^{-1\prime} & \mathcal{I}(\delta_0)^{-1} \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{E}\left[\left(2\left(\frac{G_1}{G}\right)^2 - \frac{G_2}{G}\right) tt' t_q\right] \end{pmatrix} \\
&\times \begin{pmatrix} 0 & 0 & \Gamma^{-1}\Pi_S \\ \Pi_S'\Lambda^{-1} & -\Pi_S'\left(F_\omega^{-1} + \Lambda^{-1}\Pi_0\right) & \Pi_S' F_\omega^{-1}\Pi_S \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\left(2\left(\frac{G_1}{G}\right)^2 - \frac{G_2}{G}\right) tt_q t'\right] \end{pmatrix} \\
&\times \begin{pmatrix} 0 & 0 & \Gamma^{-1}\Pi_S \\ \Pi_S'\Lambda^{-1} & -\Pi_S'\left(F_\omega^{-1} + \Lambda^{-1}\Pi_0\right) & \Pi_S' F_\omega^{-1}\Pi_S \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ \text{(Not Needed)} & \text{(Not Needed)} & \text{(Not Needed)} \end{pmatrix},
\end{aligned}
$$

the first K rows of which equal (88).

We also calculate (89) similarly. First note that, for $q = K+1+M+1, \ldots, K+2(1+M)$, we have

$$
\mathbb{E}\left[\frac{\partial^2 m_q(Z_i,\beta)}{\partial\beta\partial\beta'}\right] = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{E}\left[\frac{\partial^2}{\partial\delta\partial\delta'} S_{\delta,q-K-1-M}\right] \end{pmatrix}.
$$

Multiplying out we then get, for $q = K+1, \ldots, K+1+M$,

$$
\begin{aligned}
&\Upsilon \mathbb{E}\left[\frac{\partial^2 m_q(Z_i,\beta)}{\partial\beta\partial\beta'}\right] H\Omega L' \\
&= \begin{pmatrix} \mathcal{I}(\gamma_0)^{-1} & 0 \\ \Pi_S'\Gamma^{-1\prime} & \mathcal{I}(\delta_0)^{-1} \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{E}\left[\frac{\partial^2}{\partial\delta\partial\delta'} S_{\delta,q-K-1-M}\right] \end{pmatrix} \\
&\times \begin{pmatrix} 0 & 0 & \Gamma^{-1}\Pi_S \\ \Pi_S'\Lambda^{-1} & -\Pi_S'\left(F_\omega^{-1} + \Lambda^{-1}\Pi_0\right) & \Pi_S' F_\omega^{-1}\Pi_S \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{I}(\delta_0)^{-1}\mathbb{E}\left[\frac{\partial^2}{\partial\delta\partial\delta'} S_{\delta,q-K-1-M}\right] \end{pmatrix} \\
&\times \begin{pmatrix} 0 & 0 & \Gamma^{-1}\Pi_S \\ \Pi_S'\Lambda^{-1} & -\Pi_S'\left(F_\omega^{-1} + \Lambda^{-1}\Pi_0\right) & \Pi_S' F_\omega^{-1}\Pi_S \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ \text{(Not Needed)} & \text{(Not Needed)} & \text{(Not Needed)} \end{pmatrix},
\end{aligned}
$$

the first K rows of which give (89).

To get the overall bias expression for the class of three step AIPW estimators given in the statement to Theorem

3.1 we note that

$$
\begin{aligned}
&-\frac{\Gamma^{-1}}{N}\mathbb{E}\left[\frac{D}{G}\psi\left(\frac{D}{G}\psi-\omega\Pi_0 t_0\right)'\Lambda^{-1}\Pi_S S_\delta\right]+\frac{\Gamma^{-1}}{N}\mathbb{E}\left[\omega\left(\frac{D}{G}-\nu\right)\Pi_0 tt'\Pi_0\Lambda^{-1}\Pi_S S_\delta\right]\\
&=-\frac{\Gamma^{-1}}{N}\mathbb{E}\left[\frac{D}{G^2}\psi\psi'\Lambda^{-1}\Pi_S S_\delta\right]+\frac{\Gamma^{-1}}{N}\mathbb{E}\left[\omega\frac{D}{G}\psi t_0'\Pi_0'\Lambda^{-1}\Pi_S S_\delta\right]+\frac{\Gamma^{-1}}{N}\mathbb{E}\left[\omega\left(\frac{D}{G}-\nu\right)\Pi_0 tt'\Pi_0\Lambda^{-1}\Pi_S S_\delta\right]\\
&=-\frac{\Gamma^{-1}}{N}\mathbb{E}\left[\frac{D}{G^2}\psi\psi'\Lambda^{-1}\Pi_S S_\delta\right]+\frac{\Gamma^{-1}}{N}\mathbb{E}\left[\omega\frac{D}{G}\Pi_0 tt_0'\Pi_0'\Lambda^{-1}\Pi_S S_\delta\right]+\frac{\Gamma^{-1}}{N}\mathbb{E}\left[\omega\left(\frac{D}{G}-\nu\right)\Pi_0 tt'\Pi_0\Lambda^{-1}\Pi_S S_\delta\right]\\
&=-\frac{\Gamma^{-1}}{N}\mathbb{E}\left[\frac{D}{G^2}\psi\psi'\Lambda^{-1}\Pi_S S_\delta\right]+\frac{\Gamma^{-1}}{N}\mathbb{E}\left[\omega\left(2\frac{D}{G}-v\right)\Pi_0 tt_0'\Pi_0'\Lambda^{-1}\Pi_S S_\delta\right]\\
&=-\frac{\Gamma^{-1}}{N}\mathbb{E}\left[\frac{D}{G^2}\left\{\psi\psi'-\Pi_0 tt_0'\Pi_0'\right\}\Lambda^{-1}\Pi_S S_\delta\right]+\frac{\Gamma^{-1}}{N}\mathbb{E}\left[\left\{\omega\left(\frac{D}{G}-v\right)+\frac{D}{G}\left(\omega-\frac{1}{G}\right)\right\}\Pi_0 tt_0'\Pi_0'\Lambda^{-1}\Pi_S S_\delta\right]\\
&=-\frac{\Gamma^{-1}}{N}\mathbb{E}\left[\frac{D}{G^2}\Sigma\left(X\right)\Lambda^{-1}\Pi_S S_\delta\right]+\frac{\Gamma^{-1}}{N}\mathbb{E}\left[\left\{\omega\left(\frac{D}{G}-v\right)+\frac{D}{G}\left(\omega-\frac{1}{G}\right)\right\}\Pi_0 tt_0'\Pi_0'\Lambda^{-1}\Pi_S S_\delta\right]\\
&=-\frac{\Gamma^{-1}}{N}\mathbb{E}\left[\frac{D}{G^2}\Sigma\left(X\right)\Lambda^{-1}\Pi_S S_\delta\right]+\frac{\Gamma^{-1}}{N}\mathbb{E}\left[\left\{\frac{D}{G}\left(2\omega-\frac{1}{G}\right)-\omega v\right\}\Pi_0 tt_0'\Pi_0'\Lambda^{-1}\Pi_S S_\delta\right]
\end{aligned}
$$

Using the last equality and the expression for C_L, C_{NL1} and C_{NL2} given in the main appendix then gives the result.

References

[1] Henderson, H. V. and S. R. Searle. (1981). "On deriving the inverse of a sum of matrices," *SIAM Review* 23 (1): 53 - 60.